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Short Distance Expansion from the Dual Representation of Infinite Dimensional Lie Algebras [1]

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ABSTRACT

We compute the short distance expansion of fields or operators that live in the coadjoint representation of an infinite dimensional Lie algebra by using only properties of the adjoint representation and its dual. We explicitly compute the short distance expansion for the duals of the Virasoro algebra, affine Lie Algebras and the geometrically realized N-extended supersymmetric $\mathcal{GR}$ Virasoro algebra.

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1 Introduction

The Virasoro algebra is at the heart of understanding string theory and low dimensional gravitational theories. In string theory and conformal field theories it is often thought of as a derived quantity that comes from the mode expansion of an energy-momentum tensor. For mathematicians it also has meaning in its own right as the one dimensional algebra of centrally extended Lie derivatives. Representations of the Virasoro algebra are used to classify conformal field theories and also provide important clues as to the nature of string field theories. One representation, the coadjoint representation has been the focus of investigations for two distinct reasons. One is that the orbits of the coadjoint representation under the action of the Virasoro group have a relationship with unitary irreducible representations of the Virasoro algebra [28, 19, 34, 31]. For the Virasoro algebra and also affine Lie algebras in one dimension these orbits can then be directly related to two dimensional field theories that are conformal field theories. These are the *geometric actions* [27, 1, 9]. Another reason for studying these representations comes when one studies the elements of the coadjoint representation and adjoint representation as conjugate variables of a field theory [20]. The adjoint elements in these constructions are the conjugate momenta if they generate the isotropy algebra of the coadjoint elements. Field theories constructed in this fashion are called *transverse actions* [5, 6] (with respect to the coadjoint representation) since the geometric actions constructed on the orbits are transverse to these transverse actions. One example of the distinction of these two types of actions for the SU(N) affine Lie algebra or what physicists sometimes call an SU(N) Kac-Moody algebra is its geometric action, an SU(N) WZNW model,and its transverse action, the two dimensional SU(N) Yang-Mills action. For the Virasoro algebra the geometric action is given by the two dimensional Polyakov action for gravity [26] and its corresponding transverse action given by the $N = 0$ *affirmative action* [16, 4]. Besides these constructions, other areas of interests for the coadjoint representation are the BTZ black holes [2, 3] that appear in the asymptotic Brown-Henneaux symmetry [7] on AdS^3 [36, 24, 2].

In this note we will take this primordial view of the algebra and its dual by describing a classical phase space where the coadjoint representation provides coordinates of a phase space and the adjoint representation are the conjugate momentum variables. We will construct Noether charges on the phase space which will correspond to the adjoint action on the variables. From there it is straightforward to write the short distance expansion for the elements of the dual representation with themselves or with elements of the adjoint representation for that matter. The algebras of interests to us in this note will be the pure Virasoro algebras, the semi-direct product of the Virasoro algebra with an affine Lie Algebra, and supersymmetric extensions of this for an arbitrary number of supersymmetries. At this point a small review is in order

2 The Role of the Coadjoint Representation

In any space-time dimension the Lie algebra of coordinate transformations can be written as

$$\mathcal{L}_\xi \eta^a = -\xi^b \partial_b \eta^a + \eta^b \partial_b \xi^a = (\xi \circ \eta)^a, \tag{1}$$

and the algebra of Lie derivatives satisfies,

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{\xi \circ \eta}. \tag{2}$$

In *one* dimension, we can centrally extend this algebra by including a two cocycle which is coordinate invariant and satisfies the Jacobi identity. We write

$$[(\mathcal{L}_\xi, a), (\mathcal{L}_\eta, b)] = (\mathcal{L}_{\xi\circ\eta}, (\xi, \eta)) \tag{3}$$

where the two cocycle depends on the D-dimensional metric g_{ab} (used to define the connection) and a rank two tensor[6] D_{ab},

$$(\xi, \eta) = \tfrac{c}{2\pi} \int (\xi^a \nabla_a \nabla_b \nabla_c \eta^c)\, dx^b + \tfrac{h}{2\pi} \int (\xi^a D_{ab} \nabla_c \eta^c)\, dx^b - (\xi \leftrightarrow \eta). \tag{4}$$

Here the index structure is left in tact in order to show the invariance of the two cocycle. This expression can be viewed as residing on a 0-brane. Since this is a one dimensional structure, one may ignore the indices as long as one is mindful of the tensor structure and write,

$$(\xi, \eta) = \tfrac{c}{2\pi} \int (\xi\eta''' - \xi'''\eta)\, dx + \tfrac{h}{2\pi} \int (\xi\eta' - \xi'\eta)(D + \tfrac{c}{h}(\Gamma' + \tfrac{1}{2}\Gamma^2))\, dx, \tag{5}$$

where $'$ is the derivative with respect to the coordinate along the 0-brane. In this form, it is easy to see that (5) contains the familiar "anomaly" term that arises in string theory. The two cocycle then can be reduced to

$$(\xi, \eta) = \tfrac{c}{2\pi} \int (\xi\eta''' - \xi'''\eta)\, dx + \tfrac{h}{2\pi} \int (\xi\eta' - \xi'\eta)\, B\ dx, \tag{6}$$

dependent upon a pseudo-tensor, $B = (hD + c\,(\Gamma' + \frac{1}{2}\Gamma^2))$, which transforms as

$$\delta B = -2\xi' B - \xi B' - c\,\xi''' \tag{7}$$

under infinitesimal coordinate transformations. This pseudo tensor absorbs the metric contribution in the central extension as well as the tensor D. B is said to transform in the coadjoint representation of the Virasoro algebra. Different choices of B will give different centrally extended algebras. In string theory, central extensions are commonly chosen so that an SL(2,R) subalgebra is centerless. Thus the coadjoint representation might be thought of as endemic to the central extension of the algebra. For example, $B = 0$, is the choice commonly used when the metric is fixed to $g_{ab} = 1$ and $D_{ab} = 0$. The vector fields ξ for the one dimensional line are moded by $\xi = \sum C_N x^{N+1}$. Then the realization for the Virasoro algebra is

$$[L_N, L_M] = (N - M)\, L_{N+M} + (cN^3 - cN)\, \delta_{N+M,0}. \tag{8}$$

One can get the same central extension for the algebra on the circle in two distinct ways. Either by making a change of coordinates where $x = \exp{(i\omega)}$ giving the complex metric $g(\omega) = \exp{(2i\omega)}$ or by choosing $g(\tau) = 1$ and $D_{ab} = g_{ab}$ on the circle with $h = -c$. In this case the vector fields are $\xi = \sum\, C_N\, \exp{(iN\omega)}$.

As a parenthetic remark, one may wish to view the two cocycle (ξ, η) as a functional of the metric. Upon variation of (ξ, η) with respect to the metric g_{ab} and assuming that D_{ab} is independent of the metric, one finds

$$\frac{\delta}{\delta g}(\xi, \eta) = \mathrm{J}'\,\Gamma + \mathrm{J}\,\Gamma', \tag{9}$$

where the current is, $\mathrm{J} = \eta\,\xi' - \eta'\,\xi$. For constant J, one recognizes this variation as the anomalous one dimensional "energy-momentum" tensor. Examples of higher dimensional versions of non-central extensions may be found in [22, 23, 10].

[6]While it is possible to choose $D_{ab} = g_{ab}$, at this stage we do not impose this condition.

Another way in which the coadjoint representation appears is directly through the construction of a representation that is dual to the adjoint representation. In this case one starts with the centrally extended algebra. Then using a suitable pairing between the algebra and its dual, one extracts the coadjoint action of the algebra. As an example consider the semi-direct product of the Virasoro algebra with an affine Lie algebra. Then the Virasoro algebra

$$[L_N, L_M] = (N - M)\, L_{N+M} + cN^3\, \delta_{N+M,0} \tag{10}$$

is augmented with

$$\left[J^\alpha_N, J^\beta_M\right] = i\, f^{\alpha\beta\gamma}\, J^\beta_{N+M} + N\, k\, \delta_{N+M,0}\, \delta^{\alpha\beta} \tag{11}$$

and

$$[L_N, J^\alpha_M] = -M\ J^\alpha_{N+M}. \tag{12}$$

where $[\tau^\alpha, \tau^\beta] = i f^{\alpha\beta\gamma}\tau^\gamma$. The algebra is realized by

$$L_N = \xi^a_N \partial_a = i e^{iN\theta}\partial_\theta, \quad J^\alpha_N = \tau^\alpha e^{iN\theta}, \tag{13}$$

so that the centrally extended basis can be thought of as the three-tuple,

$$\left(L_A, J^\beta_B, \rho\right). \tag{14}$$

The adjoint representation acts on itself as

$$\left(L_A, J^\beta_B, \rho\right) * \left(L_{N'}, J^{\alpha'}_{M'}, \mu\right) = (L_{new}, J_{new}, \lambda) \tag{15}$$

where

$$\begin{aligned} L_{new} &= (A - N')\, L_{A+N'} \\ J_{new} &= -M' J^{\alpha'}_{A+M'} + B J^\beta_{B+N'} + i f^{\,\beta\alpha'\lambda} J^\lambda_{B+M'} \\ \lambda &= (cA^3)\delta_{A+N',0} + Bk\delta^{\alpha'\beta}\delta_{B+M',0}. \end{aligned} \tag{16}$$

A typical basis for the dual of the algebra can be written as the three-tuple $\left(\widetilde{L_N}, \widetilde{J^\alpha_M}, \widetilde{\mu}\right)$.

Using the pairing,

$$\left\langle \left(\widetilde{L_N}, \widetilde{J^\alpha_M}, \widetilde{\mu}\right) \middle| \left(L_A, J^\beta_B, \rho\right) \right\rangle = \delta_{N,A} + \delta^{\alpha\beta}\delta_{M,B} + \rho\widetilde{\mu} \tag{17}$$

and requiring it to be invariant one defines the coadjoint representation through the action of the adjoint on this dual as [21]

$$\left(L_A, J^\beta_B, \rho\right) \quad * \quad \left(\tilde{L}_N, \tilde{J}^\alpha_M, \tilde{\mu}\right) = \left(\tilde{L}_{new}, \tilde{J}^\alpha_M, 0\right) \qquad \text{with,} \tag{18}$$

$$\tilde{L}_{new} = \qquad (2A - N)\tilde{L}_{N-A} - B\delta^{\alpha\beta}\tilde{L}_{M-B} - \tilde{\mu}(cA^3)\tilde{L}_{-A} \qquad \text{and} \tag{19}$$

$$\tilde{J}^\alpha_M = \qquad (M - A)\tilde{J}^\alpha_{M-A} - i f^{\,\beta\nu\alpha}\tilde{J}^\nu_{M-B} - \tilde{\mu}B\, k\, \tilde{J}^\beta_{-B}. \tag{20}$$

Since we are interested in field theories it is more instructive to use explicit tensors instead of the mode decomposition. The adjoint representation may be though of as three-tuple,

$$\mathcal{F} = (\xi(\theta), \Lambda(\theta), a) \tag{21}$$

containing a vector field ξ^a, coming from the Virasoro algebra, a gauge parameter Λ coming from the affine Lie algebra and a central extension a. The coadjoint element is the three-tuple,

$$\mathrm{B} = (\mathrm{D}(\theta), \mathrm{A}(\theta), \mu), \tag{22}$$

which consists of a rank two pseudo tensor D_{ab}, a gauge field A_a and a corresponding central element μ. In this way the coadjoint action can be written as

$$\begin{aligned} \delta\widetilde{\mathrm{B}_F} &= (\xi(\theta), \Lambda(\theta), a) * (\mathrm{D}(\theta), \mathrm{A}(\theta), \mu) \\ &= (\delta\mathrm{D}(\theta), \delta\mathrm{A}(\theta), 0), \end{aligned} \tag{23}$$

$$\delta\mathrm{D}(\theta) = \underbrace{2\xi^{'}\mathrm{D} + \mathrm{D}^{'}\xi + \tfrac{c\mu}{2\pi}\xi''' + \tfrac{h\mu}{2\pi}\xi'}_{\text{coordinate transformation}} - \underbrace{Tr(\mathrm{A}\Lambda')}_{\text{gauge trans}} \tag{24}$$

and

$$\delta\mathrm{A}(\theta) = \underbrace{\mathrm{A}'\xi + \xi'\mathrm{A}}_{\text{coord trans}} - \underbrace{[\Lambda\,\mathrm{A} - \mathrm{A}\,\Lambda] + k\,\mu\,\Lambda'}_{\text{gauge transformation}}. \tag{25}$$

Again $'$ denotes derivative with respect to the argument.

3 As Phase Space Variables

The role of the adjoint and coadjoint representations as phase space elements is best represented in a familiar example with emphasis on the algebraic structure. Consider the phase space elements of a Yang-Mills theory. There one has the vector potential as a canonical coordinate, $A_i^a(x)$ and the electric field $E_i^a(x)$ as its conjugate momentum through the Poisson bracket relations

$$\begin{aligned} [A_i^a(x), A_j^b(y)] &= 0 \\ [E_i^a(x), E_j^b(y)] &= 0 \\ [A_i^a(x), E_j^b(y)] &= i\delta^{ab}\,\delta_{ij}\,\delta(x,y). \end{aligned} \tag{26}$$

The transformation laws for these phase space variables under spatially dependent gauge transformations is given by

$$\begin{aligned} A_i(x) &\rightarrow U(x)A_iU^{-1}(x) - \tfrac{1}{g}\partial_iU(x)U^{-1}(x) \\ E_i(x) &\rightarrow U(x)E_iU^{-1}(x), \end{aligned} \tag{27}$$

where one recognizes that $A_i(x)$ (when reduced on a 0-brane) is the second element of the three-tuple in Eq(22) of the element of the coadjoint representation of an SU(N) affine Lie algebra on the line and $E_i(x)$ is in the adjoint representation (the second element in Eq(21). In this way one sees that these representations which are dual to each other are indeed phase space elements.

If we were to consider the generating function for spatial gauge transformations,

$$G(x)^a = \partial_i E_i^a + [E_i, A_i]^a, \tag{28}$$

then one can construct the charge

$$Q_\Lambda \quad = \quad \int dx\; G^a\Lambda^a(x) \qquad \text{with} \tag{29}$$

$$\begin{aligned} \{Q_\Lambda, E(x)\} &= [\Lambda(x), E(x)] \\ \{Q_\Lambda, A(x)\} &= [\Lambda(x), A(x)] - \tfrac{1}{g}\partial\Lambda(x) \end{aligned} \tag{30}$$

For a generic function on the phase space, say $F(A,E)$

$$\{Q_\Lambda, F(A,E)\} = \Lambda(x)\frac{\delta}{\delta\hat{\Lambda}(x)} F(A,E) \tag{31}$$

where $\frac{\delta}{\delta\hat{\Lambda}(x)}$ is the functional variation in the direction of $\Lambda(x)$.

Similarly, the Virasoro algebra and its dual produce a set of phase space variables. Let the one dimensional pseudo tensor, D_{ij} (when reduced on a 0-brane) correspond to the first element in Eq.(22) with conjugate momentum given by the rank two tensor density of weight one, X^{ij}. The generator corresponding to the one dimensional coordinate transformation is given by

$$G_a(x) = X^{lm}\,\partial_a D_{lm} - \partial_l(X^{lm}\,D_{am}) - \partial_m(D_{la}\,X^{lm}) - c\;\partial_a\partial_l\partial_m X^{lm}. \tag{32}$$

¿From the Poisson brackets, one can recover the transformation laws of X^{ab} and D_{ab}. We have

$$Q_\xi = \int dx\;G_a\xi^a, \tag{33}$$

where ξ^a (when reduced on a 0-brane) corresponds to ξ in the algebra. One has

$$\begin{aligned} \{Q_\xi,\,D_{lm}(x)\} &= -\xi^a(x)\,\partial_a D_{lm}(x) - D_{am}(x)\,\partial_l\xi^a(x) \;-\; D_{la}(x)\,\partial_m\xi^a(x) - c\;\;\partial_a\partial_l\partial_m\,\xi^a(x) \\ &= -2\xi'(x)\,D(x) - \xi(x)\,D'(x) - c\,\xi(x)''' \\ \{Q_\xi,\,X^{lm}(x)\} &= \xi^a(x)\partial_a\,X^{lm}(x) - (\partial_a\xi^l(x))\,X^{am}(x) - (\partial_a\xi^m(x))\,X^{la}(x) + \,(\partial_a\xi^a(x))X^{lm}(x) \\ &= \xi(x)\,X'(x) - \xi'(x)\,X(x) \end{aligned} \tag{34}$$

Again for a generic function on the phase space, say $F(D,X)$

$$\{Q_\xi, F(D,X)\} = \xi(x)\,\frac{\delta}{\delta\hat{\xi}(x)}\,F(D,X) \tag{35}$$

where $\frac{\delta}{\delta\hat{\xi}(x)}$ is the functional variation in the direction of $\xi(x)$.

4 Short Distance Expansions from Algebras

It is now straightforward to extract the short distance expansion [33] for elements of the coadjoint representation. Equations (31) and (35) are the prototype expressions that we need to proceed. For any element of the adjoint representation, say α, we can construct Q_α. Now α is dual to coadjoint elements, say $\mathcal{A}$ since

$$(\alpha,\mathcal{A}) = \int \alpha(x)\mathcal{A}(x)\,dx = \quad \text{some constant.} \tag{36}$$

We treat the above expression as the integral form of the operation with $\frac{\delta}{\delta\hat{\alpha}}$. Then for any function of the phase space $F(x)$

$$\{Q_\alpha, F(x)\} \;=\; \alpha(x)\,\frac{\delta}{\delta\hat{\alpha}(x)}\,F(x) \tag{37}$$

$$= \int \alpha(y)\,(\mathcal{A}(y)G(x))\;dy. \tag{38}$$

The distribution $f_A(x,y) = \mathcal{A}(y)F(x)$ when compared to the left hand side of Eq(38) gives the short distance expansion between $\mathcal{A}(x)$ and $F(x)$. In the following we derive short distance expansions for some simple cases and then move on to the N-extended super Virasoro algebra.

4.1 Case: Virasoro and Affine Lie Algebras

As our first example we consider the short distance expansion for the semi direct product of the Virasoro algebra and an affine Lie algebra. We are interested in the short distance expansion of the coadjoint elements $D(x)$ and $A(x)$. Equations 24 and 25 allow us to construct the charge

$$Q_\xi = \int X(y) \left(2\xi'(y)\, D(y) + D(y)'\, \xi + \frac{ic\mu}{12}\, \xi''' \right) + \int E(y) \left(\xi'(y) A(y) + A'(y) \xi(y) \right). \tag{39}$$

Then from $\{Q_\xi,\, A(x)\}$ we have that

$$\{Q_\xi,\, A^b(x)\} = \xi(x) \frac{\delta}{\delta \hat{\xi}(x)} A^b(x) = \int \xi(y)\, D(y)\, A^b(x)\, dy\ . \tag{40}$$

This implies that

$$D(y)\, A^b(x) = (\partial_y \delta(x,y))\, A^b(x) - (\partial_x A^b(x))\, \delta(x,y). \tag{41}$$

Using the representation for the delta function on a line

$$\delta(x,y) = \frac{1}{2\pi i(y-x)}, \tag{42}$$

we have that

$$D(y)\, A^b(x) = \frac{-1}{2\pi i (y-x)^2}\, A^b(x) - \frac{1}{2\pi i(y-x)} \partial_x A^b(x). \tag{43}$$

Similarly we can construct a charge Q_Λ via

$$Q_\Lambda = \int \left(-X(y) A^b(y) \Lambda'^b(y) - i f^{bac} E^b(y) \Lambda^a(y) A^c(y) + k\mu\, E^b(y)\, \Lambda'^b(y) \right)\, dy \tag{44}$$

then together Q_L and Q_ξ will give

$$D(y)\, D(x) = \frac{-1}{2\pi i(y-x)} \partial_x D(x) + \frac{1}{\pi i (y-x)^2} D(x) - \frac{c}{2\pi i (y-x)^4} \tag{45}$$

$$D(y)\, A^b(x) = \frac{-1}{2\pi i(y-x)^2} A^b(x) - \frac{1}{2\pi i(y-x)} \partial_x A^b(x) \tag{46}$$

$$A^b(y)\, D(x) = \frac{1}{(y-x)^2}\, A^b(x) \tag{47}$$

$$A^b(y)\, A^a(x) = \frac{ik\mu}{(y-x)^2} \delta^{ba} - i\, f^{bac} \frac{1}{(y-x)} A^c(x). \tag{48}$$

4.2 Case: N=1 Super Virasoro Algebras

Supersymmetric algebras can be treated in a similar way. From the discussion found in Siegel [29], let Φ^M represent an element of the phase space and let Ω^{MN} be a supersymplectic two form with an inverse Ω_{MN}, i.e.

$$\Omega^{MN}\, \Omega_{PN} = \delta^M_P.$$

The index $M = \{m, \mu\}$ where the Latin indices are bosonic and the Greek indices fermionic. Then

$$[\Phi^M, \Phi^N\} = \hbar\, \Omega^{MN}, \tag{49}$$

and

$$\Omega_{\{MN]} = 0.$$

This means that

$$\Omega_{(mn)} = \Omega_{[\mu\nu]} = \Omega_{m\nu} + \Omega_{\nu m} = 0.$$

The bracket is defined by

$$[A, B\} = -A \, \frac{\overleftarrow{\partial}}{\partial \Phi^M} \, \Omega^{NM} \, \frac{\partial B}{\partial \Phi^N} \tag{50}$$

For the N=1 super Virasoro algebra, the adjoint representation is built from the vector field ξ, a spinor field ϵ and and a central extension so that we might write an adjoint element as $\mathcal{F} = (\xi, \epsilon, a)$. In the same way the coadjoint representation is given by a three-tuple of fields, $B = (D, \psi, \alpha)$. The transformation law for the coadjoint representation is

$$\mathcal{F} * B = \left(-\xi \, D' - \tfrac{1}{2} \epsilon \, \psi' - \tfrac{3}{2} \epsilon' \psi - \tfrac{c\,\beta}{8} \, \xi''', \, -\xi \, \psi' - \tfrac{1}{2} \epsilon \, \psi \, - \tfrac{3}{2} \xi' \, \psi - 4 i \beta c \, \epsilon'', \, 0 \right). \tag{51}$$

The charges are

$$Q_\xi \;=\; \int X(y) \left(-\xi(y) \, D(y) \; - \tfrac{c\,\beta}{8} \, \xi''' \right) dy + \int \phi(y) \left(-\xi \, \psi(y)' - \tfrac{3}{2} \xi' \, \psi(y) \right) \, dy, \tag{52}$$

and

$$Q_\epsilon \;=\; \int X(y) \left(-\tfrac{1}{2} \epsilon(x) u(y)' - \tfrac{3}{2} \epsilon'(y) \, \psi(y) \right) \, dy + \int \phi(y) \left(-\tfrac{1}{2} \epsilon(y) - 4 i \beta c \, \epsilon'' \right) \, dy \tag{53}$$

Here the fields $X(y)$ and $\phi(y)$ correspond respectively to the spin -1 and spin $-\frac{1}{2}$ conjugate momenta for D and ψ, which gives the short distance expansion

$$D(y) \, D(x) \;=\; \frac{-1}{2\pi i (y-x)} \partial_x D(x) + \frac{1}{\pi i (y-x)^2} D(x) - \frac{3 c \beta}{8 \pi i (y-x)^4} \tag{54}$$

$$D(y) \, \psi(x) \;=\; \frac{-3}{4 \pi i (y-x)^2} \psi(x) - \frac{1}{2 \pi i (y-x)} \partial_x \psi(x) \tag{55}$$

$$\psi(y) \, D(x) \;=\; \frac{3}{4 \pi i (y-x)^2} \, \psi(x) - \frac{1}{4 \pi i (y-x)} \psi(x) \tag{56}$$

$$\psi(y) \, \psi(x) \;=\; \frac{-2}{\pi (y-x)} D(x) - \frac{4 \beta c}{\pi (y-x)^3}. \tag{57}$$

5 N-Extended $\mathcal{GR}$ Super Virasoro Algebra

We now come to the final example and the main result of this work, the N-extended Super Virasoro algebras. The N Extended super Virsoro algebra that we will use is the one found ing [14, 15]. The generators of this algebra provide an (almost[8]) primary basis to the $K(1 \mid N)$ contact superalgebra [7]. This is the subalgebra of vect$(1 \mid N)$ vector fields that preserves the contact one form

$$\sigma = d\tau + \delta_{IJ} \zeta^I d\zeta^J.$$

The generators are

$$\mathcal{Q}_{\mathcal{A}}^{\mathrm{I}_1 \cdots \mathrm{I}_\mathrm{p}} \;=\; \tau^{\mathcal{A}} \zeta^{\mathrm{I}_1} \cdots \zeta^{\mathrm{I}_\mathrm{p}} \partial_\tau \quad , \quad \mathcal{P}_{\mathcal{A}}^{\mathrm{I}_1 \cdots \mathrm{I}_{\mathrm{p}+1}} \;=\; \tau^{\mathcal{A}} \zeta^{\mathrm{I}_1} \cdots \zeta^{\mathrm{I}_\mathrm{p}} \partial^{\mathrm{I}_{\mathrm{p}+1}} \quad ,$$

where $\mathcal{A}$ can be integer or half-integer moded. This algebra along with a complete classification of Lie superalgebras used in string theory are reviewed and discussed in [17]. In the "almost" primary

[7] We thank Thomas Larsson for pointing this out to us.

basis the generators may be written as [8]: the following:

$$G_{\mathcal{A}}{}^{\rm I} \;\; \equiv \;\; i\,\tau^{\mathcal{A}+\frac{1}{2}}\left[\,\partial^{\rm I} \;-\; i\,2\,\zeta^{\rm I}\,\partial_\tau\,\right] \;+\; 2(\,\mathcal{A}\,+\,\tfrac{1}{2}\,)\tau^{\mathcal{A}-\frac{1}{2}}\zeta^{\rm I}\zeta^{\rm K}\,\partial_{\rm K} \quad , \tag{58}$$

$$L_{\mathcal{A}} \;\; \equiv \;\; -\left[\,\tau^{\mathcal{A}+1}\partial_\tau \;+\; \tfrac{1}{2}(\mathcal{A}\,+\,1)\,\tau^{\mathcal{A}}\zeta^{\rm I}\,\partial_{\rm I}\,\right] \quad , \tag{59}$$

$$U_{\mathcal{A}}^{{\rm I}_1\cdots{\rm I}_q} \;\; \equiv \;\; i\,(i)^{[\frac{q}{2}]}\,\tau^{(\mathcal{A}-\frac{(q-2)}{2})}\zeta^{{\rm I}_1}\cdots\,\zeta^{{\rm I}_{q-1}}\,\partial^{{\rm I}_q} \quad , \quad q \,=\, 1,\,\dots\,,\,N\,+\,1 \quad , \tag{60}$$

$$\mathcal{R}_{\mathcal{A}}^{{\rm I}_1\cdots{\rm I}_p} \;\; \equiv \;\; (i)^{[\frac{p}{2}]}\,\tau^{(\mathcal{A}-\frac{p}{2})}\zeta^{{\rm I}_1}\cdots\,\zeta^{{\rm I}_p}\,\tau\partial_\tau \quad , \quad p \,=\, 2,\,\dots\,,\,N \tag{61}$$

for any number N of supersymmetries. The antisymmetric part of the $U_{\mathcal{A}}^{[{\rm I\,J}]}$ generator will generate an SO(N) subalgebra so we will denote it as

$$T_{\mathcal{A}}^{\rm I\,J} \equiv -U_{\mathcal{A}}^{[{\rm I\,J}]}.$$

Note that the primary genrators $L_{\mathcal{A}}$ and $G_{\mathcal{A}}{}^{\rm I}$ are used in lieu of $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{A}}^{{\rm I}_1}$ respectively while $\mathcal{R}_{\mathcal{A}}^{{\rm I}_1{\rm I}_2}$ is the only generator that is not primary.

The elements of the algebra can be realized as fields whose tensor properties can be determined by they way they transform under one dimensional coordinate transformations. How each field transforms under a Lie derivative with respect to ξ is summarized below. In conformal field theory these transformation laws are generalized for two copies of the diffeomorphism algebra and characterize the fields by weight and spin. Here we treat the algebraic elements as one dimensional tensors.

Table 1: Tensors Associated with the Algebra

Element of algebra	Transformation Rule	Tensor Structure
$L_{\mathcal{A}} \;\to\; \eta$	$\eta \;\to\; -\xi'\eta + \xi\eta'$	η^a
$G^{\rm I}_{\mathcal{A}} \;\to\; \chi^{\rm I}$	$\chi^{\rm I} \;\to\; -\xi(\chi^{\rm I})' + \frac{1}{2}\xi'\chi^{\rm I}$	$\chi^{{\rm I};\alpha}$
$T^{\rm RS} \;\to\; t^{\rm RS}$	$t^{\rm RS} \;\to\; -\,\xi\,(t^{\rm RS})'$	$t^{\rm RS}$
$U^{{\rm V}_1\cdots{\rm V}_{\rm n}} \;\to\; w^{{\rm V}_1\cdots{\rm V}_{\rm n}}$	$w^{{\rm V}_1\cdots{\rm V}_{\rm n}} \to -\xi(w^{{\rm V}_1\cdots{\rm V}_{\rm n}})' - \frac{1}{2}(n-2)\xi' w^{{\rm V}_1\cdots{\rm V}_{\rm n}}$	$w^{{\rm V}_1\cdots{\rm V}_{\rm n};\,{\rm a}}_{\qquad\alpha_1\cdots\alpha_n}$
$R^{{\rm T}_1\cdots{\rm T}_{\rm n}}_{\mathcal{A}} \;\to\; r^{{\rm T}_1\cdots{\rm T}_{\rm n}}$	$r^{{\rm T}_1\cdots{\rm T}_{\rm n}} \;\to\; -\,(r^{{\rm T}_1\cdots{\rm T}_{\rm n}})'\xi - \frac{1}{2}(n-2)\xi' r^{{\rm T}_1\cdots{\rm T}_{\rm n}}$	$r^{{\rm T}_1\cdots{\rm T}_{\rm n}\,;a}_{\qquad\alpha_1\cdots\alpha_n}$

In the above table we have used capital Latin letters, such as I, J, K to represent SO(N) indices, small Latin letters to represent tensor indices, and small Greek letters for spinor indices. Spinors with their indices up transform as scalar tensor densities of weight one (1) while those with their indices down transform as scalar densities of weight minus one (-1). For example, the generator $U^{{\rm V}_1{\rm V}_2{\rm V}_3}$ has a tensor density realization of contravariant tensor with rank one and weight $-\frac{3}{2}$ living in the N × N × N representation of SO(N), i.e. $\omega^{{\rm V}_1{\rm V}_2{\rm V}_3;a}_{\alpha_1\alpha_2\alpha_3}$.

Table 2: Tensors Associated with the Dual of the Algebra

Dual element of algebra	Transformation Rule	Tensor Structure
$L^{\star}{}_{\mathcal{A}} \;\to\; D$	$D \;\to\; -\,2\xi' D - \xi D'$	D_{ab}
$G^{\star}{}^{\rm I}_{\mathcal{A}} \;\to\; \psi^{\rm I}$	$\psi^{\rm I} \;\to\; -\xi(\psi^{\rm I})' - \frac{3}{2}\xi'\psi^{\rm I}$	$\psi^{\rm I}_{a\alpha}$
$T^{\star\,{\rm RS}} \;\to\; A^{\rm RS}$	$A^{\rm RS} \;\to\; -\,(\xi)' A^{\rm RS} - \xi\,(A^{\rm RS})'$	$A^{\rm RS}_a$
$U^{\star\,{\rm V}_1\cdots{\rm V}_{\rm n}} \;\to\; \omega^{{\rm V}_1\cdots{\rm V}_{\rm n}}$	$\omega^{{\rm V}_1\cdots{\rm V}_{\rm n}} \;\to\; -\,\xi(\omega^{{\rm V}_1\cdots{\rm V}_{\rm n}})' - (2-\frac{n}{2})\xi'\omega^{{\rm V}_1\cdots{\rm V}_{\rm n}}$	$\omega^{{\rm V}_1\cdots{\rm V}_{\rm n};\,\alpha_1\cdots\alpha_{\rm n}}_{ab}$
$R^{\star\,{\rm T}_1\cdots{\rm T}_{\rm r}}_{\mathcal{A}} \;\to\; \rho^{{\rm T}_1\cdots{\rm T}_{\rm r}}$	$\rho^{{\rm T}_1\cdots{\rm T}_{\rm r}} \;\to\; -\,(\rho^{{\rm T}_1\cdots{\rm T}_{\rm r}})'\xi - (2-\frac{r}{2})\xi'\rho^{{\rm T}_1\cdots{\rm T}_{\rm r}}$	$\rho^{{\rm T}_1\cdots{\rm T}_{\rm r};\alpha_1\cdots\alpha_r}_{ab}$

[8]We are grateful to Thomas Larsson for pointing out errors in the algebra in previous publications

Thus for N supersymmetries there is one rank two tensor D_{ab}, N spin-$\frac{3}{2}$ fields ψ^{I}, a spin-1 covariant tensor A^{RS} that serves as the $N(N-1)/2$ SO(N) vector potentials (that are gauge potentials for $N = 2$) associated with the supersymmetries, $N\,(2^N)$ fields for the $\omega^{\mathrm{V_1\cdots V_p}}$ and $N\,(2^N - N - 1)$ $\rho^{\mathrm{T_1\cdots T_p}}$ fields. The entries in the third column of each of these tables are the 0-brane reduced tensors or tensor densities corresponding to the ones that appear in the second column.

5.1 Central Extensons of the N-Extended GR algebra

In [8, 4] it was mistakenly reported that the N-Extended GR algebra admits a central extension for arbitrary values of N and further the $U_{\mathcal{A}}^{I_1}$ and the symmetric combination for $U_{\mathcal{A}}^{(I_1 I_2)}$ generators where omitted. It is easy to see that not only are these required to close the algebra but their existence will eliminate the central extensions for $N > 2$. Although the algebra is not simple since the $U_{\mathcal{A}}^{I}$ fields form an abelian extension to the algebra the obstruction comes from the required symmetic part of $U_{\mathcal{A}}^{J_1 J_2}$. This can be seen by looking at the $(G_{\mathcal{A}}^{I}, U_{\mathcal{B}}^{IJ}, G_{\mathcal{C}}^{K})$ contribution to the Jacobi identity. There one finds that

$$[U_{\mathcal{B}}^{J_1 J_2}, \{G_{\mathcal{A}}^{I}, G_{\mathcal{C}}^{K}\}] - \{G_{\mathcal{C}}^{K}, [U_{\mathcal{B}}^{J_1 J_2}, G_{\mathcal{A}}^{I}]\} + \{G_{\mathcal{A}}^{I}, [G_{\mathcal{C}}^{K}, U_{\mathcal{B}}^{J_1 J_2}]\} = 0$$

implies that

$$-2i(A - C)\,[U_{\mathcal{B}}^{J_1 J_2},\, U_{\mathcal{A}+\mathcal{C}}^{IK}] - i\tilde{c}\left((C^2 - \tfrac{1}{4})\delta^{J_1 K}\delta^{I J_2} - (A^2 - \tfrac{1}{4})\delta^{J_2 K}\delta^{I j_1}\right)\delta_{A+B+C} = 0.$$

It is clear that the symmtric combination cannot satisfy the Jacobi identity. As we will show below, only the antisymmtric fields $U_{\mathcal{A}}^{[I_1 I_2]}$ are needed to close the algbra and a central extension exists. The constraint above forces the central extension in the SO(2) affine Kac-Moody algebra to be related to the central extension in the superdiffeomorphism algebra even before unitarity issues are considered. In what follows we will seperate the $N = 2$ algebra from the generic case.

5.2 $N = 2$

For $N = 2$ the $\mathcal{GR}$ Super Virasoro Algebra reduces to a smaller set of generators that close as well as admits a central extension. The algebra is

$$[\, L_{\mathcal{A}}\,,\, L_{\mathcal{B}}\,\} \;=\; (\,\mathcal{A} - \mathcal{B}\,)\, L_{\mathcal{A}+\mathcal{B}} + \tfrac{1}{8}\, c\,(\mathcal{A}^3 - \mathcal{A})\,\delta_{\mathcal{A}+\mathcal{B},0} \quad , \tag{62}$$

$$\begin{aligned}[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, G_{\mathcal{B}}{}^{\mathrm{J}}\,\} \;=\;& -i\,4\,\delta^{\mathrm{IJ}} L_{\mathcal{A}+\mathcal{B}} \;-\; i2(\mathcal{A}-\mathcal{B})\,[\, T_{\mathcal{A}+\mathcal{B}}^{\mathrm{IJ}} \;+\; 2(\mathcal{A}+\mathcal{B})\, U_{\mathcal{A}+\mathcal{B}\mathrm{K}}^{\mathrm{IJK}}\,] \\ & -ic(\mathcal{A}^2 - \tfrac{1}{4})\,\delta_{\mathcal{A}+\mathcal{B},0}\,\delta^{\mathrm{IJ}} \quad ,\end{aligned} \tag{63}$$

$$[\, L_{\mathcal{A}}\,,\, G_{\mathcal{B}}{}^{\mathrm{I}}\,\} \;=\; (\,\tfrac{1}{2}\mathcal{A} - \mathcal{B}\,)\, G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{I}} \quad , \tag{64}$$

$$[L_{\mathcal{A}}\,,\, T_{\mathcal{B}}^{\mathrm{IJ}}\} \;=\; -\,\mathcal{B}\, T_{\mathcal{A}+\mathcal{B}}^{\mathrm{IJ}} \quad , \tag{65}$$

$$\begin{aligned}[\, T_{\mathcal{A}}^{\mathrm{IJ}}\,,\, T_{\mathcal{B}}^{\mathrm{KL}}\,\} \;=\;& T_{\mathcal{A}+\mathcal{B}}^{\mathrm{IK}}\,\delta^{\mathrm{JL}} + T_{\mathcal{A}+\mathcal{B}}^{\mathrm{JL}}\,\delta^{\mathrm{IK}} - T_{\mathcal{A}+\mathcal{B}}^{\mathrm{IL}}\,\delta^{\mathrm{JK}} - T_{\mathcal{A}+\mathcal{B}}^{\mathrm{JK}}\,\delta^{\mathrm{IL}} \\ & -2c\,(\mathcal{A}-\mathcal{B})(\delta^{\mathrm{IK}}\delta^{\mathrm{JL}} - \delta^{\mathrm{IL}}\delta^{\mathrm{JK}})\delta_{\mathcal{A}+\mathcal{B},0} \quad ,\end{aligned} \tag{66}$$

$$[\, T_{\mathcal{A}}^{\mathrm{I\,J}}\,,\, G_{\mathcal{B}}{}^{\mathrm{K}}\,\} \quad = \quad 2\,(\delta^{\mathrm{JK}} G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{I}} - \delta^{\mathrm{IK}} G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{J}}) \tag{67}$$

Where

$$T_{\mathcal{A}}^{\mathrm{I\,J}} \equiv -U_{\mathcal{A}}^{[\mathrm{I\,J}]}$$

serves as the SO(2) generator.

5.3 $N > 2$

The $N > 2$ the central extension is absent and new generators must be included including the "low order" generators $U_{\mathcal{A}}^{\mathrm{I}}$. The commutation relations are

$$[\, L_{\mathcal{A}}\,,\, L_{\mathcal{B}}\,\} \quad = \quad (\,\mathcal{A} - \mathcal{B}\,)\, L_{\mathcal{A}+\mathcal{B}} \tag{68}$$

$$[\, L_{\mathcal{A}}\,,\, U_{\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m}\,\} \quad = \quad -\,[\,\mathcal{B} + \tfrac{1}{2}\,(m-2)\,\mathcal{A}\,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m} \quad , \tag{69}$$

$$[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, G_{\mathcal{B}}{}^{\mathrm{J}}\,\} \quad = \quad -\,i\,4\,\delta^{\mathrm{I\,J}} L_{\mathcal{A}+\mathcal{B}} \;-\; i2(\mathcal{A}-\mathcal{B})\,[\, T_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}} \;+\; 2(\mathcal{A}+\mathcal{B})\, U_{\mathcal{A}+\mathcal{B}\,\mathrm{K}}^{\mathrm{I\,J\,K}}\,] \;\; , \tag{70}$$

$$[\, L_{\mathcal{A}}\,,\, G_{\mathcal{B}}{}^{\mathrm{I}}\,\} \quad = \quad (\,\tfrac{1}{2}\mathcal{A} - \mathcal{B}\,)\, G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{I}} \quad , \tag{71}$$

$$[\, L_{\mathcal{A}}\,,\, R_{\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m}\,\} \quad = \quad -\,[\,\mathcal{B} + \tfrac{1}{2}\,(m-2)\,\mathcal{A}\,]\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m} - \;[\,\tfrac{1}{2}\,\mathcal{A}\,(\mathcal{A}+1)\,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m\,J}{}_{J} \quad , \tag{72}$$

$$\begin{aligned}[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, R_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_m}\,\} \quad = \quad & 2\,(i)^{\sigma(m)}\,[\,\mathcal{B} + (m-1)\,\mathcal{A} + \tfrac{1}{2}\,]\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}_1\cdots\mathrm{J}_m} \\ & - (i)^{\sigma(m)} \sum_{r=1}^{m} (-1)^{r-1}\, \delta^{I\,\mathrm{J}_r}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{r-1}\,\mathrm{J}_{r+1}\cdots\mathrm{J}_m} \\ & - (-i)^{\sigma(m)}\,[\,\mathcal{A} + \tfrac{1}{2}\,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_m\,\mathrm{I}} \\ & + 2\,(i)^{\sigma(m)}\,[\,\mathcal{A}^2 - \tfrac{1}{4}\,]\, U_{\mathcal{A}+\mathcal{B}}^{I\,\mathrm{J}_1\cdots\mathrm{J}_m\,K}{}_{\mathrm{K}} \quad , \qquad m \neq 2\end{aligned} \tag{73}$$

$$\begin{aligned}[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, R_{\mathcal{B}}^{\mathrm{J}_1\mathrm{J}_2}\,\} \quad = \quad & 2(\mathcal{A}+\mathcal{B}+\tfrac{1}{2}) R_{\mathcal{A}+\mathcal{B}}^{\mathrm{IJ}_1\mathrm{J}_2} - (\mathcal{A}-\tfrac{1}{2})\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\mathrm{J}_2\mathrm{I}} + 2(\mathcal{A}^2-\tfrac{1}{4})\, U_{\mathcal{A}+\mathcal{B}\;\mathrm{K}}^{\mathrm{IJ}_1\mathrm{J}_2\mathrm{K}} \\ - \quad & \delta^{IJ_1}\left(\tfrac{1}{2} G_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_2} - U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_2} + 2(\mathcal{A}+\mathcal{B}+\tfrac{1}{2}) U_{\mathcal{A}+\mathcal{B}\;K}^{\mathrm{J}_2\mathrm{K}}\right) \\ + \quad & \delta^{IJ_2}\left(\tfrac{1}{2} G_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1} - U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1} + 2(\mathcal{A}+\mathcal{B}+\tfrac{1}{2}) U_{\mathcal{A}+\mathcal{B}\;K}^{\mathrm{J}_1\mathrm{K}}\right)\end{aligned} \tag{74}$$

$$\begin{aligned}[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, U_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_m}\,\} \quad = \quad & 2\,(i)^{\sigma(m)}\,[\,\mathcal{B} + (m-2)\,\mathcal{A}\,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}_1\cdots\mathrm{J}_m} \\ & - 2\,(-i)^{\sigma(m)}\,[\,\mathcal{A} + \tfrac{1}{2}\,]\, \delta^{\mathrm{I\,J}_m}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{m-1}K}{}_{K} \\ & - (i)^{\sigma(m)} \sum_{r=1}^{m-1} (-1)^{r-1}\, \delta^{\mathrm{I\,J}_r}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{r-1}\,\mathrm{J}_{r+1}\cdots\mathrm{J}_m} \\ & + 2\,(-i)^{\sigma(m)}\, \delta^{\mathrm{I\,J}_m}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{m-1}} \quad , \qquad m \neq 2\end{aligned} \tag{75}$$

$$[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, U_{\mathcal{B}}^{\mathrm{J}}\,\} \quad = \quad -2i\,\delta^{\mathrm{IJ}}(L_{\mathcal{A}+\mathcal{B}} + \tfrac{1}{2}(\mathcal{B}-\mathcal{A})\, U_{\mathcal{A}+\mathcal{B}\,\mathrm{K}}^{\mathrm{K}}) - 2i\,(\mathcal{A}+\mathcal{B}+1)\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{IJ}} \tag{76}$$

$$[\, G_{\mathcal{A}}{}^{\mathrm{I}}\,,\, U_{\mathcal{B}}^{\mathrm{J}_1\mathrm{J}_2}\,\} \quad = \quad -U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_2}\, \delta^{\mathrm{IJ}_1} + 2\mathcal{B}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{IJ}_1\mathrm{J}_2} + \delta^{\mathrm{J}_2\mathrm{I}}\left(G_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1} - U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1} + 2\mathcal{B}\, U_{\mathcal{A}+\mathcal{B}\,\mathrm{K}}^{\mathrm{J}_1\mathrm{K}}\right) \tag{77}$$

$$[\, R_{\mathcal{A}}^{\mathrm{I}_1\cdots\mathrm{I}_m}\,,\, R_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_n}\,\} \quad = \quad -\,(i)^{\sigma(mn)}\,[\,\mathcal{A} - \mathcal{B} - \tfrac{1}{2}(m-n)\,]\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m\,\mathrm{J}_1 n\cdots\mathrm{J}_n} \quad , \tag{78}$$

$$[\, R_{\mathcal{A}}^{\mathrm{I}_1\cdots\mathrm{I}_m} \,,\, U_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_n} \,\} \;=\; (-i)^{\sigma(mn)} \sum_{r=1}^{m} (-1)^{r-1}\, \delta^{\mathrm{I}_r\,\mathrm{J}_n}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{n-1}\mathrm{I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_m}$$
$$+\; i(i)^{\sigma(mn)}\,[\,\mathcal{B} - \tfrac{1}{2}(n-2)\,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_m\,\mathrm{J}_1\cdots\mathrm{J}_n}, \quad m \neq 2,\, n \neq 1 \tag{79}$$

$$\begin{aligned}[\, R_{\mathcal{A}}^{\mathrm{I}_1\mathrm{I}_2} \,,\, U_{\mathcal{B}}^{\mathrm{J}} \,\} \;=\;& (\mathcal{B}+\tfrac{1}{2}) U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\mathrm{I}_2\mathrm{J}} \\ +\;& \tfrac{1}{2}\big(U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_2} - G_{\mathcal{A}+\mathcal{B}}^{I_2} + 2(\mathcal{A}+\mathcal{B}+\tfrac{1}{2})\, U_{\mathcal{A}+\mathcal{B}\ \mathrm{K}}^{\mathrm{I}_2\mathrm{K}}\big)\delta^{\mathrm{J}_1\mathrm{I}_1} \\ -\;& \tfrac{1}{2}\big(U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1} - G_{\mathcal{A}+\mathcal{B}}^{I_2} + 2(\mathcal{A}+\mathcal{B}+\tfrac{1}{2})\, U_{\mathcal{A}+\mathcal{B}\ \mathrm{K}}^{\mathrm{I}_1\mathrm{K}}\big)\delta^{\mathrm{J}_1\mathrm{I}_2}\end{aligned} \tag{80}$$

$$[\, U_{\mathcal{A}}^{\mathrm{I}_1\cdots\mathrm{I}_m} \,,\, U_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_n} \,\} \;=\; -\,(i)^{\sigma(mn)} \Big\{ \sum_{r=1}^{m} (-1)^{r-1}\, \delta^{\mathrm{I}_m\mathrm{J}_r}\; U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_{m-1}\,\mathrm{J}_1\cdots\mathrm{J}_{r-1}\,\mathrm{J}_{r+1}\cdots\mathrm{J}_{n-1}\mathrm{J}_n}$$
$$-(-1)^{mn} \;\textstyle\sum_{r=1}^{m}\; (-1)^{r-1}\, \delta^{\mathrm{I}_r\mathrm{J}_n}\; U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{n-1}\,\mathrm{I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_{m-1}\mathrm{I}_m} \Big\}, \quad n \neq 1 \tag{81}$$

$$[\, U_{\mathcal{A}}^{\mathrm{I}_1} \,,\, U_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_m} \,\} \;=\; -(i)^{\sigma(m)}\, \{\sum_{r=1}^{n}(-1)^{r-1}\, \delta^{\mathrm{I}_1\mathrm{J}_r}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\hat{\mathrm{J}}_r\cdots\mathrm{J}_n}. \tag{82}$$

where the function $\sigma(m) = 0$ if m is even and -1 if m is odd. The central extensions c and $\tilde{c}$ are unrelated since we have only imposed the Jacobi identity.

5.4 Short Distance Expansion for $D(y)\,O(x)$

The transformation laws for $\xi(y)$ on the dual of the algebra allows us to extract the short distance expansion rules for the field $D(y)$ and any other element in the coadjoint representation $O(x)$. Using the transformation rules

$$L_\xi * (\bar{L}_D, \bar{\beta}) \;=\; \bar{L}_{\tilde{D}} \quad , \quad \tilde{D} \;=\; -\,2\,\xi' D \,-\, \xi\, D' \,-\, \tfrac{c\bar{\beta}}{8}\, \xi'''\, \delta_{N,2} \quad , \tag{83}$$

$$L_\xi * \bar{G}_{\Psi^{\bar{Q}}}^{\bar{Q}} \;=\; \bar{G}_{\tilde{\Psi}^{\bar{Q}}}^{\bar{Q}} \quad , \quad \tilde{\Psi^{\bar{Q}}} \;=\; -\,(\tfrac{3}{2}\xi'\psi^{\bar{Q}} + \xi(\psi^{\bar{Q}})') \quad , \tag{84}$$

$$L_\xi * \bar{T}_{\tau^{\bar{\mathrm{R}}\bar{\mathrm{S}}}}^{\bar{\mathrm{R}}\bar{\mathrm{S}}} \;=\; \bar{T}_{\tilde{\tau}^{\bar{\mathrm{R}}\bar{\mathrm{S}}}}^{\bar{\mathrm{R}}\bar{\mathrm{S}}} \quad , \quad \tilde{\tau}^{\bar{\mathrm{R}}\bar{\mathrm{S}}} \;=\; -\xi'\tau^{\bar{\mathrm{R}}\bar{\mathrm{S}}} \,-\, \xi\,(\tau^{\bar{\mathrm{R}}\bar{\mathrm{S}}})' \quad , \tag{85}$$

$$L_\xi * \bar{U}_{\omega^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}}}^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}} \;=\; \bar{U}_{\tilde{\omega}^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}}}^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}} \,+\, \tfrac{i}{2}(i)^{[\frac{n-2}{2}]-[\frac{n}{2}]}\, \bar{R}_{\xi''\omega^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}}}^{[\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_{\mathrm{n}-2}}\,\delta^{\mathrm{V}_{\mathrm{n}-1}]\mathrm{V}_\mathrm{n}} \quad , \tag{86}$$

$$\text{where} \quad \tilde{\omega}^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}} \;=\; (\tfrac{n}{2}-2)\,\xi'\,\omega^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}} \,-\, \xi(\omega^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_\mathrm{n}})' \quad ,$$

$$L_\xi * \bar{R}_{\rho^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}}}^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}} \;=\; \bar{R}_{\tilde{\rho}^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}}}^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}} \quad , \tag{87}$$

$$\text{where} \quad \tilde{\rho}^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}} \;=\; (\tfrac{\mathrm{m}}{2}-2)\,\xi'\,\rho^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}} - \xi\,(\rho^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_\mathrm{m}})'$$

and $[x]$ is the greatest integer in x.

By constructing the generators we have the short distance expansion laws,

$$D(y)D(x) \;=\; \frac{-1}{2\pi(y-x)}\,\partial_x D(x) - \frac{1}{\pi i(y-x)^2}\, D(x) - \frac{3\,c\,\beta\,\delta_{N,2}}{4\pi i(y-x)^4} \tag{88}$$

$$D(y)\,\psi^Q(x) \;=\; \frac{-3}{4\pi i(y-x)^2}\,\psi^Q(x) - \frac{1}{2\pi i(y-x)}\partial_x\psi^Q(x) \tag{89}$$

$$D(y)\, A^{RS}(x) \;=\; \frac{-1}{2\pi i(y-x)^2}\, A^{RS}(x) - \frac{1}{2\pi i(y-x)}\partial_x A^{RS}(x) \tag{90}$$

$$D(y)\, \omega^{V_1\cdots V_n}(x) \;=\; \frac{n-4}{4\pi i(y-x)^2}\, \omega^{V_1\cdots V_n}(x) - \frac{1}{2\pi i(y-x)}\partial_x \omega^{V_1\cdots V_n}(x) \tag{91}$$

$$\begin{aligned} D(y)\, \rho^{V_1\cdots V_n}(x) \;&=\; \frac{n-4}{4\pi i(y-x)^2}\, \rho^{V_1\cdots V_n}(x) - \frac{1}{2\pi i(y-x)}\partial_x \rho^{V_1\cdots V_n}(x) \\ &+\; \frac{(i)^{[\frac{n}{2}]-[\frac{n-2}{2}]}}{2\pi(y-x)^3}\omega_A^{I_1\cdots I_n A}(x) \end{aligned} \tag{92}$$

It is worth commenting on the last summand that appears in Eq.(92). From Eq.(86), we notice that the transformation of $\omega^{V_1\cdots V_n}(x)$ contributes to $\rho^{I_1\cdots I_m}(x)$. This implies that the charge Q_ξ contains the summand

$$Q_\xi = \cdots + \; i(i)^{[\frac{q-2}{2}]-[\frac{q}{2}]}\frac{1}{2}\int \Pi_\rho^{I_1\cdots I_{q-2}}(x)\left(\xi''(x)\omega^{I_1\cdots I_{q-2}I_{q-1}I_q}(x)\right)\delta_{q-1,q}\, dx, \tag{93}$$

where $\Pi_\rho^{I_1\cdots I_{q-2}}(x)$ is the conjugate momentum for $\rho^{I_1\cdots I_{q-2}}(x)$. So

$$\{Q_\xi, \rho^{I_1\cdots I_{q-2}}(x)] = \xi(x)\frac{\delta}{\delta\xi(x)}\rho^{I_1\cdots I_{q-2}}(x), \tag{94}$$

will lead to

$$D(y)\, \rho^{I_1\cdots I_n}(x) = \tfrac{i}{2} i^{[\frac{n}{2}]-[\frac{n-2}{2}]}\, \partial_y^2\delta(y,x)\, \omega_A^{I_1\cdots I_n A}(x) + (\text{plus}\, \rho^{I_1\cdots I_n}\ \text{dependent terms}). \tag{95}$$

The existence of contributions like ξ'' as seen in Eq.(86) is due to the transformation of the connection Γ. Although the connection does not appear in the algebra since $g_{ab} = 1$, its presence is felt in the coordinate transformations.

5.5 Short Distance Expansion for $\psi^I(y)\, O(x)$

The next sections follow in the same way as the $D(y)\, O(x)$ expansions. One constructs the generators dedicated to the symmetry transformation of the particular algebraic element and then identifies the short distance expansion or operator product expansion from the variation. From [8] we can write the transformation laws due to the spin $\frac{1}{2}$ fields as

$$\begin{aligned} G^{\mathrm{I}}_{\chi^{\mathrm{I}}} * \bar{G}^{\bar{\mathrm{Q}}}_{\psi^{\bar{\mathrm{Q}}}} \;&=\; \delta^{\mathrm{I}\bar{\mathrm{Q}}}\, \bar{L}_{\tilde{\xi}} + \bar{T}^{\mathrm{I}\bar{\mathrm{Q}}}_{\tilde{A}^{\mathrm{I}\bar{\mathrm{Q}}}} + (1-\delta_2^N)\left(\bar{R}^{\mathrm{I}\bar{\mathrm{Q}}}_{\chi^{\mathrm{I}}\psi^{\bar{\mathrm{Q}}}} - \bar{U}^{\mathrm{I}\bar{\mathrm{Q}}}_{\chi^{\mathrm{I}}\psi^{\bar{\mathrm{Q}}}}\right) \\ \text{where} \qquad & \tilde{\xi} = \tfrac{1}{2}\,(\psi^{\bar{\mathrm{Q}}})'\chi^I - \tfrac{3}{2}\,(\chi^I)'\psi^{\bar{\mathrm{Q}}}, \quad \tilde{A}^{\mathrm{I}\bar{\mathrm{Q}}} = (2\chi^{\mathrm{I}}\psi^{\bar{\mathrm{Q}}} - 2\chi^{\mathrm{Q}}\psi^{\bar{\mathrm{I}}}) \end{aligned} \tag{96}$$

$$\begin{aligned} G^{\mathrm{I}}_{\chi^{\mathrm{I}}} * (\bar{L}_D, \bar{\beta}) \;&=\; 4i\, \bar{G}^{\mathrm{I}}_{\tilde{\chi}^{\mathrm{I}}} - 2i\tilde{U}^{\mathrm{I}}_{\chi^{\mathrm{I}}D}, \\ \text{where} \qquad & \tilde{\chi}^{\mathrm{I}} = (-\chi^{\mathrm{I}}\, D - \bar{\beta}c\, \delta_2^N\, (\chi^{\mathrm{I}})'') \end{aligned} \tag{97}$$

$$G^{\mathrm{I}}_{\chi^{\mathrm{I}}} * \bar{T}^{\bar{\mathrm{R}}\mathrm{S}}_{\tau^{\bar{\mathrm{R}}\mathrm{S}}} \;=\; \tfrac{i}{2}(\bar{G}^{\mathrm{S}}_{\chi^{\mathrm{S}}}\, \delta^{\bar{\mathrm{R}}\mathrm{I}} - \bar{G}^{\bar{\mathrm{R}}}_{\chi^{\bar{\mathrm{R}}}}\, \delta^{\mathrm{I}\bar{\mathrm{S}}}) \quad , \quad \chi^{\bar{\mathrm{R}}} \;=\; \chi^{\mathrm{S}} = 2(\chi^{\mathrm{I}})'\tau^{\bar{\mathrm{R}}\,\mathrm{S}} + \chi^{\mathrm{I}}\,(\tau^{\bar{\mathrm{R}}\,\mathrm{S}})' \tag{98}$$

$$G^{\mathrm{I}}_{\chi^{\mathrm{I}}} * \bar{U}^{\mathrm{V_1V_2}}_{\omega^{\mathrm{V_1V_2}}} \;=\; -i\bar{U}^{\mathrm{I}}_{(-2\chi^{\mathrm{I}\prime}\omega^{\mathrm{V_1V_2}} - \omega^{\mathrm{V_1V_2}\prime}\chi^{\mathrm{I}})}\, \delta^{\mathrm{V_1V_2}} + 2i\bar{U}^{\mathrm{V_2}}_{(\omega^{\mathrm{V_1V_2}\prime}\chi^{\mathrm{I}})}\delta^{\mathrm{V_1I}}, \qquad N \neq 2 \tag{99}$$

$$G^{\rm I}_{\chi^{\rm I}} * \bar{R}^{\rm LM} \;=\; -3i\,\bar{R}^{\rm ILM}_{(\chi^{\rm I}\rho^{\rm LM})} \tag{100}$$

$$\begin{aligned} G^{\rm I}_{\chi^{\rm I}} * \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_m}_{\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_m}} \;=\;& 2i(i)^{m+1}(i)^{[\frac{m+2}{2}]-[\frac{m}{2}]}\,\bar{U}^{[\bar{\rm T}_1\cdots\bar{\rm T}_m]}_{(\chi^{\rm I}\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_m})} \qquad m\neq 2 \\ & -2i^{[\frac{m-1}{2}]-[\frac{m-2}{2}]}\,\delta^{I\,[\bar{T}_1}\,\bar{R}^{\bar{\rm T}_2\cdots\bar{\rm T}_m]}_{((\chi^{\rm I})'\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_m}-(\chi^{\rm I})(\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_m})')} \\ & -(i)(i)^{[\frac{m+1}{2}]-[\frac{m}{2}]}\sum_{r=1}^{m+1}(-1)^{r-1}\,\bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_{r-1}\,{\rm I}\,\bar{\rm T}_{r+1}\cdots\bar{\rm T}_m}_{(\chi^{\rm I}\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_m})} \end{aligned} \tag{101}$$

$$\begin{aligned} G^{\rm I}_{\chi^{\rm I}} * \bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_n}_{\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n}} \;=\;& -2i^{[\frac{n-1}{2}]-[\frac{n}{2}]}\,\delta^{I\,[\bar{V}_1}\,\bar{U}^{\bar{\rm V}_2\cdots\bar{\rm V}_n]}_{((n-4)(\chi^{\rm I})'\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n}-(\chi^{\rm I})(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})')} \\ & +2(-1)^{n-1}(i)^{[\frac{n-1}{2}]-[\frac{n}{2}]}\,\delta^{\rm I[\bar{V}_n}\bar{U}^{\bar{V}_1\cdots\bar{V}_{n-1}]\,{\rm K}}_{((\chi^{\rm I})'\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})^{\rm K}} \\ & +(i)(i)^{[\frac{n-1}{2}]-[\frac{n}{2}]}\sum_{r=1}^{n}\bar{U}^{[\bar{\rm V}_1\cdots\bar{\rm V}_{r-1}\bar{\rm V}_{r+1}\cdots\bar{\rm V}_n}_{(\chi^{\rm I}\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})}\,\delta^{\bar{\rm V}_r]\,{\rm I}} \\ & +\bar{G}^{[\bar{\rm V}_2}_{(-4i\,(\chi^{\rm I})'(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})'-2i\,(\chi^{\rm I})(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})'')}\,\delta^{\bar{\rm V}_3\bar{\rm V}_4}\,\delta^{\bar{\rm V}_1]{\rm I}}\,\delta^{n4} \\ & -(-1)^{n-1}\,\delta^{\rm I[\bar{V}_n}\,\bar{R}^{\bar{\rm V}_1\cdots\bar{\rm V}_{n-1}]}_{((\chi^{\rm I})'\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})} \\ & -2\,\delta^{\rm I[\bar{V}_1}\,\bar{R}^{\bar{\rm V}_2\cdots\bar{\rm V}_{n-2}}_{((\chi^{\rm I})''\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n})}\,\delta^{{\rm V}_{n-1}]{\rm V}_n} \quad n\neq 1,2 \end{aligned} \tag{102}$$

$$G^{\rm I}_{\chi^{\rm I}} * \bar{U}^{\bar{\rm V}_1}_{\omega^{\bar{\rm V}_1}} \;=\; -\bar{R}^{\rm IV_1}_{(\chi^{\rm I}\omega^{\rm K})} - \bar{U}^{\rm IK}_{(\chi^{\rm I}\omega^{\rm K})} + \bar{U}^{\rm KI}_{(\omega^{\rm K}\chi^{\rm I})} \tag{103}$$

The short distance expansions that follow from here are:

$$\psi^I(y)\,D(x) \;=\; \frac{-i}{4\pi(y-x)}\,\partial_x(\psi^I(x)) - \frac{3}{4\pi i(y-x)^2}\,\psi^I(x) \tag{104}$$

$$\psi^I(y)\,\psi^Q(x) \;=\; \frac{-4i}{(y-x)}\,\delta^{IQ}\,D(x) - \delta^N_2\,\frac{8i\beta c}{(y-x)^2}\delta^{IQ} \tag{105}$$

$$\psi^A(y)\,A^{RS}(x) \;=\; \frac{\pi}{i(y-x)}\,\left(\delta^{AR}\delta^{LS}-\delta^{AS}\delta^{LR}\right)\,\psi^L(x) \tag{106}$$

$$\begin{aligned} \psi^J(y)\,\omega^{A_1\cdots A_{q-1}}_{A_q}(x) \;=\;& (1-\delta^q_2)\frac{1}{2\pi i(y-x)}2(i)(i)^q\,(i)^{[\frac{q+1}{2}]-[\frac{q-1}{2}]}\,\delta^J_{A_q}\,\rho^{A_1\cdots A_{q-1}}(x) \\ & - \frac{i^{[\frac{q}{2}]-[\frac{q+1}{2}]}}{\pi i(y-x)}\left(\frac{(q-3)}{(y-x)}\omega^{JA_1\cdots A_{q-1}}_{A_q}(x) - \partial_x\omega^{JA_1\cdots A_{q-1}}_{A_q}(x)\right) \\ & + (1-\delta^q_2)(-1)^{(q-2)}\,(-i)^{[\frac{q-2}{2}]-[\frac{q-1}{2}]}\frac{1}{\pi i(y-x)^2}\,\omega^{[A_1\cdots A_{q-2}J}\delta^{A_{q-1}]}_{A_q}(x) \\ & + \frac{(i)^{[\frac{q-1}{2}]-[\frac{q}{2}]}}{2\pi i(y-x)}\sum_{r=1}^{q}\delta^{A_1}_{[V_1}\cdots\delta^{A_{r-1}}_{V_{r-1}}\delta^{I}_{V_r}\cdots\delta^{A_{q-1}}_{V_q}\delta_{V_{q+1}]A_q}\,\omega^{V_1\cdots V_{q+1}}(x) \\ & - \frac{i\delta^q_3(1-\delta^N_2)}{2\pi(y-x)}\,\delta^{\rm J[A_1}\,\psi^{\rm A_2]}(x) \\ & - i\frac{\delta^q_2(1-\delta^N_2)}{2\pi(y-x)}\delta^{\rm J[A_1}\,\omega^{\rm A_2]}(x) \end{aligned} \tag{107}$$

$$\psi^J(y)\,\rho^{A_1\cdots A_q}(x) \;=\; \frac{-(-1)^{(q+1)}}{2\pi(y-x)}(i)^{[\frac{q+1}{2}]-[\frac{q}{2}]}\omega^{A_1\cdots A_qJ}(x) + \frac{i\delta^q_2(1-\delta^N_2)}{2\pi(y-x)}\,\delta^{\rm J[A_1}\,\psi^{\rm A_2]}(x)$$

$$
\begin{aligned}
\frac{-(-1)^{(q-1)}}{2\pi(y-x)}(i)^{[\frac{q+1}{2}]-[\frac{q}{2}]} \quad & \sum_{r=1}^{q+1} \delta^{A_1}_{[V_1} \cdots \delta^{A_{r-1}}_{V_{r-1}} \delta^{A_r}_{I} \delta^{A_{r+1}}_{V_r} \cdots \delta^{A_q}_{V_{q-1}]} \, \delta^{IJ} \, \rho^{V_1 \cdots V_{q-1}}(x)(1-\delta^q_2) \\
- \quad & \frac{i^{[\frac{q-1}{2}]-[\frac{q-2}{2}]}}{\pi i(y-x)} \left(\frac{1}{(y-x)} \, \rho^{JA_1 \cdots A_q} - \partial_x \rho^{JA_1 \cdots A_q} \right) \\
- \quad & i \frac{\delta^q_2 (1-\delta^N_2)}{2\pi(y-x)} \delta^{\mathrm{J}[\mathrm{A}_1} \, \omega^{\mathrm{A}_2]}(x)
\end{aligned} \tag{108}
$$

5.6 Short Distance Expansion for $A^{JK}(y)\, O(x)$

As a generator of SO(N) transformations, the operator $A^{JK}(y)$ takes on its own significance from the symmertic part of $U^{JK}(y)$. Here are some of the expansions for operators $O(x)$ paired with $A^{JK}(y)$.

$$
T^{\mathrm{J\,K}}_{t^{\mathrm{J\,K}}} * \bar{G}^{\bar{\mathrm{Q}}}_{\psi^{\bar{\mathrm{Q}}}} \quad = \quad -\,2(\bar{G}^{\mathrm{K}}_{(t^{\mathrm{J\,K}}\,\psi^{\bar{\mathrm{Q}}})} \delta^{\bar{\mathrm{Q}}\mathrm{J}} - \bar{G}^{\mathrm{J}}_{(t^{\mathrm{J\,K}}\,\psi^{\bar{\mathrm{Q}}})} \delta^{\bar{\mathrm{Q}}\mathrm{K}}) \quad , \tag{109}
$$

$$
\begin{aligned}
T^{\mathrm{J\,K}}_{t^{\mathrm{J\,K}}} * \bar{U}^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}}_{\omega^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}}} \quad = & \quad -\sum_{r=1}^{n-1} (-1)^{r+1} (\delta^{\mathrm{J}\,[\,\bar{\mathrm{V}}_1} \, \bar{U}^{\bar{\mathrm{V}}_2 \cdots \bar{\mathrm{V}}_{\mathrm{r}-1}\,|\,\mathrm{K}\,|\bar{\mathrm{V}}_{\mathrm{r}+1}\cdots]\bar{\mathrm{V}}_\mathrm{n}}_{(t^{\mathrm{J\,K}}\,\omega^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}})} \\
- & \quad \delta^{\mathrm{K}\,[\,\bar{\mathrm{V}}_1} \, \bar{U}^{\bar{\mathrm{V}}_2 \cdots \bar{\mathrm{V}}_{\mathrm{r}-1}\,|\,\mathrm{J}\,|\bar{\mathrm{V}}_{\mathrm{r}+1}\cdots]\bar{\mathrm{V}}_\mathrm{n}}_{(t^{\mathrm{J\,K}}\,\omega^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}})}) \\
+ & \quad \bar{U}^{[\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_{\mathrm{n}-1}]\,\mathrm{J}}_{(t^{\mathrm{J\,K}}\,\omega^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}})} \, \delta^{\bar{\mathrm{V}}_\mathrm{n}\,\mathrm{K}} - \bar{U}^{[\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_{\mathrm{n}-1}]\,\mathrm{K}}_{(t^{\mathrm{J\,K}}\,\omega^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}})} \, \delta^{\bar{\mathrm{V}}_\mathrm{n}\,\mathrm{J}} \\
-i & \quad (-1)^{n-2} \, (\delta^{\mathrm{K}\,\bar{\mathrm{V}}_\mathrm{n}} \, \delta^{J\,[\bar{\mathrm{V}}_1} \, - \, \delta^{\mathrm{J}\,\bar{\mathrm{V}}_\mathrm{n}} \, \delta^{\mathrm{K}\,[\bar{\mathrm{V}}_1}) \bar{R}^{\bar{\mathrm{V}}_2 \cdots \bar{\mathrm{V}}_{\mathrm{n}-1}]}_{((t^{\mathrm{J\,K}})'\,\omega^{\bar{\mathrm{V}}_1 \cdots \bar{\mathrm{V}}_\mathrm{n}})} \quad ,
\end{aligned} \tag{110}
$$

$$
\begin{aligned}
T^{\mathrm{J\,K}}_{t^{\mathrm{J\,K}}} * \bar{R}^{\bar{\mathrm{T}}_1 \cdots \bar{\mathrm{T}}_\mathrm{m}}_{\rho^{\bar{\mathrm{T}}_1 \cdots \bar{\mathrm{T}}_\mathrm{m}}} \quad = & \quad \sum_{r=1}^{m} (-1)^{r+1} \, (\delta^{[\bar{\mathrm{T}}_1|\,\mathrm{J}\,|} \, \bar{R}^{\bar{\mathrm{T}}_2 \cdots \bar{\mathrm{T}}_{\mathrm{r}-1}\,|\,\mathrm{K}\,|\,\bar{\mathrm{T}}_{\mathrm{r}+1}\cdots\bar{\mathrm{T}}_\mathrm{m}]}_{(t^{\mathrm{J\,K}}\,\rho^{\bar{\mathrm{T}}_1 \cdots \bar{\mathrm{T}}_\mathrm{m}})} \\
& \quad -\delta^{[\bar{\mathrm{T}}_1|\,\mathrm{K}\,|} \, \bar{R}^{\bar{\mathrm{T}}_2 \cdots \bar{\mathrm{T}}_{\mathrm{r}-1}\,|\,\mathrm{J}\,|\,\bar{\mathrm{T}}_{\mathrm{r}+1}\cdots\bar{\mathrm{T}}_\mathrm{m}]}_{(t^{\mathrm{J\,K}}\,\rho^{\bar{\mathrm{T}}_1 \cdots \bar{\mathrm{T}}_\mathrm{m}})}) \quad ,
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
T^{\mathrm{J\,K}}_{t^{\mathrm{J\,K}}} * (\bar{T}^{\bar{\mathrm{R}}\bar{\mathrm{S}}}_{\tau^{\bar{\mathrm{R}}\bar{\mathrm{S}}}}, \bar{\beta}) \quad = & \quad \tfrac{1}{2}(\delta^{\bar{\mathrm{R}}\mathrm{J}}\delta^{\bar{\mathrm{S}}\mathrm{K}} - \delta^{\bar{\mathrm{R}}\mathrm{K}}\delta^{\bar{\mathrm{S}}\mathrm{J}}) \, \bar{L}_{((t^{\mathrm{JK}})'\,\tau^{\bar{\mathrm{R}}\,\bar{\mathrm{S}}})} + \tfrac{1}{2}\bar{T}^{\mathrm{A\,B}}_{(t^{\mathrm{J\,K}}\,\tau^{\bar{\mathrm{R}}\bar{\mathrm{S}}})} \, \delta^{\mathrm{JK}\bar{\mathrm{R}}\bar{\mathrm{S}}}_{\mathrm{AB}} + 4\bar{\beta}\bar{T}^{\mathrm{JK}}_{(\tau^{\bar{\mathrm{R}}\bar{\mathrm{S}}})'} \quad , \\
\text{where} \quad \delta^{\mathrm{JK}\bar{\mathrm{R}}\bar{\mathrm{S}}}_{\mathrm{AB}} \quad \equiv & \quad (\, \delta^{\mathrm{AK}}\delta^{\mathrm{B}\bar{\mathrm{S}}}\delta^{\bar{\mathrm{R}}\mathrm{J}} - \delta^{\mathrm{AK}}\delta^{\mathrm{B}\bar{\mathrm{R}}}\delta^{\bar{\mathrm{S}}\mathrm{J}} + \delta^{\mathrm{A}\bar{\mathrm{S}}}\delta^{\mathrm{BJ}}\delta^{\bar{\mathrm{R}}\mathrm{K}} \\
& \quad -\delta^{\mathrm{A}\bar{\mathrm{R}}}\delta^{\mathrm{JS}}\delta^{\bar{\mathrm{S}}\mathrm{K}} + \delta^{\mathrm{A}\bar{\mathrm{S}}}\delta^{\mathrm{BK}}\delta^{\bar{\mathrm{R}}\mathrm{J}} - \delta^{\mathrm{A}\bar{\mathrm{R}}}\delta^{\mathrm{KB}}\delta^{\bar{\mathrm{S}}\mathrm{J}} \\
& \quad +\delta^{\mathrm{AJ}}\delta^{\mathrm{B}\bar{\mathrm{S}}}\delta^{\bar{\mathrm{R}}\mathrm{K}} - \delta^{\mathrm{JA}}\delta^{\bar{\mathrm{R}}\mathrm{B}}\delta^{\bar{\mathrm{S}}\mathrm{K}} + \delta^{\mathrm{A}\bar{\mathrm{S}}}\delta^{\mathrm{BK}}\delta^{\bar{\mathrm{R}}\mathrm{J}} \,) \quad ,
\end{aligned} \tag{112}
$$

We get the $A^{JK}(y)\, D(x)$ short distance expansion as:

$$
A^{JK}(y)\, D(x) \quad = \quad \frac{1}{4\pi i(y-x)^2} \, \left(\delta^{RS}\delta^{JK} - \delta^{RK}\delta^{SJ}\right) \, A_{RS}(x) \tag{113}
$$

$$
A^{AB}(y)\, \psi^C(x) \quad = \quad \frac{-1}{\pi i(y-x)} \left(\delta^{AC}\psi^B(x) - \delta^{BC}\psi^A(x)\right) \tag{114}
$$

$$
A^{JK}(y)\, A^{RS}(x) \quad = \quad \frac{1}{4\pi i(y-x)} \delta^{JKRS}_{AB} \, A^{AB}(x) + \frac{\beta}{\pi i(y-x)^2} \left(\delta^{JR}\delta^{KS} - \delta^{RK}\delta^{SJ}\right) \tag{115}
$$

$$
\begin{aligned}
A^{JK}(y)\,\omega^{A_1\cdots A_n}(x) &= \sum_{r=1}^{n-1}\frac{(-1)^{r+1}}{2\pi(y-x)}\left(\delta^{KA_r}\,\omega^{JA_1\cdots\hat{A}_r A_{n-1}A_n}(x)-\delta^{JA_r}\omega^{KA_1\cdots\hat{A}_r A_{n-1}A_n}(x)\right)\\
&+ \frac{1}{2\pi i(y-x)}\left(\delta^{A_n J}\,\omega^{A_1\cdots A_{n-1}K}(x)-\delta^{A_n K}\,\omega^{A_1\cdots A_{n-1}J}(x)\right) \qquad (116)\\
A^{JK}(y)\,\rho^{A_1\cdots A_n}(x) &= \sum_{r=1}^{n}\frac{(-1)^{r+1}}{2\pi(y-x)}\left(\delta^{KA_r}\,\rho^{JA_1\cdots\hat{A}_r A_{n-1}A_n}(x)-\delta^{JA_r}\rho^{KA_1\cdots\hat{A}_r A_{n-1}A_n}(x)\right)\\
&+ \frac{i(-i)^{n-1}}{2\pi i(y-x)^2}\left(\omega^{JA_1\cdots A_n K}(x)-\omega^{KA_1\cdots A_n J}(x)\right) \qquad (117)
\end{aligned}
$$

5.7 Short Distance Expansion for $\omega^{I_1\cdots I_q}(y)\,O(x)$

The next two sections provide the short distance expansions for the fields $\omega^{I_1\cdots I_q}$ and $\rho^{P_1\cdots P_q}$. These fields appear in the $\mathcal{GR}$ realization as natural Grassmann extensions of the SO(N) gauge field and the one dimensional diffeomorphism field. The relevant transformation laws are :

$$U^{\rm J}_{\mu^{\rm J}} * \bar{L}_D = -2\,i\,\bar{G}^{\rm J}_{(\mu^{\rm J}\,D)} \qquad (118)$$

$$U^{\rm J}_{\mu^{\rm J}} * \bar{U}^{\rm K}_{\omega^{\rm K}}, = \bar{R}^{\rm JK}_{(\mu^{\rm J}\omega^{\rm K})} \qquad (119)$$

$$U^{\rm J}_{\mu^{\rm J}} * \bar{G}^{\rm K}_{\psi^{\rm K}} = -\bar{R}^{\rm JK}_{(\mu^{\rm J}\,\psi^{\rm K})} \qquad (120)$$

$$U^{\rm J}_{\mu^{\rm J}} * \bar{U}^{\rm LMN}_{\omega^{\rm LMN}} = \bar{R}^{\rm LM}_{(\mu'\omega)}\,\delta^{\rm JN} + 4\bar{R}^{\rm JL}_{(\mu\omega')}\,\delta^{\rm MN} \qquad (121)$$

$$U^{\rm J_1J_2}_{\mu^{\rm J_1J_2}} * \bar{U}^{\rm K}_{\omega^{\rm K}} = \bar{G}^{\rm J_1}_{(\omega^{\rm J_1J_2}\,\omega^{\rm K})}\,\delta^{\rm K\,J_2} - \bar{G}^{\rm J_2}_{(\mu^{\rm J_1J_2}\,\omega^{\rm K})}\,\delta^{\rm KJ_1} \qquad (122)$$

$$U^{\rm J_1J_2}_{\mu^{\rm J_1J_2}} * \bar{G}^{\rm K}_{\psi^{\rm K}} = -\delta^{\rm J_1K}\,\bar{G}^{\rm J_2}_{(\upsilon^{\rm J_1J_2}\,\psi^{\rm K})} \qquad (123)$$

$$
\begin{aligned}
U^{\rm I_1\cdots I_q}_{\mu^{\{\rm I_q\}}} * \bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_m}_{\omega^{\{\rm V_m\}}} &= -\delta^{\rm m\,q}\,\delta^{[\,\rm I_1\cdots I_q\,]}_{[\bar{\rm V}_1\cdots\bar{\rm V}_q\,]}\,\bar{L}_{((\frac{4-q}{2})\mu'\omega-(\frac{q-2}{2})\mu\omega')} - 2\,\bar{T}^{\bar{\rm V}_m{\rm I_q}}_{(\mu\omega)}\,\delta^{[\,\rm I_1\cdots I_{q-1}\,]}_{[\bar{\rm V}_1\cdots\bar{\rm V}_{q-1}\,]}\,\delta^{\rm q}_{\rm m}\\
&- 2i^{([\frac{q}{2}]-[\frac{q+1}{2}])}\,\delta^{\rm m,(q+1)}\,\bar{G}^{[\bar{\rm V}_1}_{(-(q-2)\mu\omega'+(3-q)\mu'\omega)}\,\delta^{\bar{\rm V}_2\cdots\bar{\rm V}_m\,]}_{[\rm I_1\cdots I_q]}\\
&+ 2(-1)^q(i)^{[\frac{q}{2}]-[\frac{q+1}{2}]}\bar{G}^{\rm I_q}_{(-\mu\omega)'}\,\delta^{\rm m,q+1}\,\delta^{[\bar{\rm V}_1\cdots\bar{\rm V}_{m-2}}_{[\rm I_1\cdots I_{q-1}]}\,\delta^{\bar{\rm V}_{m-1}],\bar{\rm V}_m}\\
&- i(i)^{[\frac{q}{2}]-[\frac{q-1}{2}]}\sum_{r=1}^{q-1}(-1)^{r-1}\delta^{\rm m}_{\rm q-1}\,\bar{G}^{[\rm I_r}_{(\mu\omega)}\delta^{\rm I_1}_{[\bar{\rm V}_1}\cdots\delta^{\rm I_{r-1}}_{\bar{\rm V}_{r-1}}\,\delta^{\rm I_{r+1}}_{\bar{\rm V}_r}\cdots\delta^{\rm I_q]}_{\bar{\rm V}_m]}\\
&+ \sum_{r=1}^{q-1}2(-1)^{r+1}(\bar{T}^{\rm I_r[\bar{V}_1}_{(\mu\omega)}\delta^{\bar{V}_2}_{[\rm I_1}\cdots\delta^{\bar{V}_r}_{\rm I_{r-1}}\,\delta^{\bar{V}_{r+1}}_{\rm I_{r+1}}\cdots\delta^{]\bar{V}_m}_{]\rm I_q}\,\delta^{\rm q,m})\\
&+ i(i)^{\{[\frac{q}{2}]+[\frac{m-q}{2}+2]-[\frac{m+2}{2}]\}}\\
&\times \{\sum_{r=1}^{m-q}(-1)^{r-1}\delta^{[\bar{\rm V}_1\cdots\bar{\rm V}_{q-1}}_{[\,\rm I_1\cdots I_{q-1}\,]}\;\bar{U}^{\bar{\rm V}_q\cdots\bar{\rm V}_{q+r-1}{\rm I_q}\bar{\rm V}_{q+r}\cdots]{\rm V}_m}_{\mu\omega}\\
&- (-1)^{q(m-q+2)}\sum_{r=1}^{q}(-1)^{r-1}\bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_{m-q+1}[\,\rm I_r}_{\mu\omega}\delta^{\rm I_1\cdots I_{r-1}I_{r+1}\cdots\,I_q]}_{\bar{\rm V}_{m-q+2}\cdots\bar{\rm V}_{m-q+2+r}\cdots\bar{\rm V}_m}\;\}
\end{aligned}
$$

$$- \quad (i)^{\{[\frac{q}{2}]+[\frac{m-q}{2}]-[\frac{q+m-4}{2}]\}} \, \bar{R}^{[\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_{\mathrm{m-q}}}_{(\mu\omega)'} \, \delta^{\bar{\mathrm{V}}_{\mathrm{m-q+1}}\cdots]\bar{\mathrm{V}}_{\mathrm{m}}}_{[\mathrm{I}_1\cdots]\mathrm{I}_{\mathrm{q}}} \quad , \tag{124}$$

$$U^{\mathrm{I}_1\cdots\mathrm{I}_{\mathrm{q}}}_{\mu^{\{\mathrm{I}_{\mathrm{q}}\}}} * \bar{R}^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{m}}}_{\rho^{\{\mathrm{T}_{\mathrm{m}}\}}} = -i(-1)^{q(m-q+2)}(i)^{\{[\frac{m-q}{2}+2]+[\frac{q}{2}]-[\frac{m}{2}]\}}$$

$$\times \sum_{r=1}^{m-q+2} (-1)^{r-1}\delta^{[\mathrm{I}_1\cdots\mathrm{I}_{\mathrm{q-1}}]}_{[\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{q-1}}]}\bar{R}^{\bar{\mathrm{T}}_{\mathrm{q}}\cdots\bar{\mathrm{T}}_{\mathrm{q+r-1}}\,\mathrm{I}_{\mathrm{q}}\,\bar{\mathrm{T}}_{\mathrm{q+r+1}}\cdots\bar{\mathrm{T}}_{\mathrm{m}}}_{\mu\rho} \tag{125}$$

For $q > 2$ the short distance expansions are:

$$\omega^{I_1\cdots I_q}(y)\,D(x) = -\frac{1}{2\pi i(y-x)}\left(\frac{4-q}{2(y-x)}\omega^{I_1\cdots I_q}(x) - \partial_x\omega^{I_1\cdots I_q}(x)\right)$$

$$\begin{aligned}
\omega^{I_1\cdots I_q}(y)\,\psi^C(x) &= \frac{(-2i)^{([\frac{q}{2}]-[\frac{q+1}{2}])}}{2\pi i(y-x)}\left(-(q-2)\partial_x\omega^{CI_1\cdots I_q}(x) + \frac{(3-q)}{(y-x)}\omega^{CI_1\cdots I_q}(x)\right) \\
&- \frac{(-1)^q(i)^{[\frac{q}{2}]-[\frac{q+1}{2}]}}{\pi i(y-x)}\delta^{CI_q}\left(\partial_x\omega_I^{I_1\cdots I_{q-1}I}(x) + \frac{1}{(y-x)}\omega_I^{I_1\cdots I_q I}(x)\right) \\
&- \frac{i(i)^{([\frac{q}{2}]-[\frac{q-1}{2}])}}{2\pi i(y-x)}\sum_{r=1}^{q-1}\delta^{c[I_r}\,\omega^{I_1\cdots\hat{I}_r\cdots I_q]}(x)
\end{aligned} \tag{126}$$

$$\begin{aligned}
\omega^{I_1\cdots I_q}(y)\,A^{AB}(x) &= \frac{1}{2\pi i(y-x)}\sum_{r=1}^{q-1}\{\omega^{AI_1\cdots\hat{I}_r\cdots I_q}(x)\,\delta^{I_rB} - \omega^{BI_1\cdots\hat{I}_r\cdots I_q}(x)\,\delta^{I_rA}\} \\
&- \frac{1}{2\pi i(y-x)}\left(\omega^{I_1\cdots I_{q-1}A}(x)\,\delta^{I_qB} + \omega^{I_1\cdots I_{q-1}B}(x)\,\delta^{I_qA}]\right)
\end{aligned} \tag{127}$$

$$\begin{aligned}
\omega^{A_1\cdots A_q}(y)\,\omega^{B_1\cdots B_p}(x) &= \frac{(i)^{[\frac{q}{2}]+[\frac{p-2}{2}+2]-[\frac{p+q}{2}]}}{2\pi(y-x)}\sum_{r=1}^{q}\omega^{V_1\cdots V_{q+r-1}BV_{q+r}\cdots v_{q+p-1}}(x)\;\times \\
&\times\; \delta^{A-1}_{[V_1}\cdots\delta^{A_{q-1}}_{V_{q-1}}\delta^{B_1}_{V_q}\cdots\delta^{B_r}_{V_{q+r-1}}\delta^{A_q}_{B}\,\delta^{B_{r+1}}_{V_{q+r}}\cdots\delta^{B_{p-1}}_{V_{q+p-2}]}\,\delta^{B_p}_{V_{q+p-1}} \\
&+\; \frac{-(-1)^{q(p+q-1)}}{2\pi i(y-x)}\sum_{r=1}^{q}\omega^{V_1\cdots V_{p+q-3}}(x)\;\times \\
&\times\; \delta^{B_1}_{V_1}\cdots\delta^{B_{p-1}}_{V_{p-1}}\delta^{B_p[A_r}\delta^{A_1}_{V_p}\cdots\delta^{A_{r-1}}_{V_{p+r-1}}\delta^{A_{r+1}}_{V_{p+r}}\cdots\delta^{A_q]}_{V_{p+q-3}}
\end{aligned} \tag{128}$$

$$\begin{aligned}
\omega^{A_1\cdots A_q}(y)\,\rho^{B_1\cdots B_p}(x) &= -(i)^{[\frac{q}{2}]+[\frac{p}{2}]-[\frac{2q+p-4}{2}]}\left(\frac{\omega^{B_1\cdots B_pA_1\cdots A_q}(x)}{2\pi i(y-x)^2} + \frac{\partial_x\omega^{B_1\cdots B_pA_1\cdots A_q}(x)}{2\pi i(y-x)}\right) \\
&+ \sum_{r=1}^{p+2}\frac{(-1)^{r-1}}{2\pi i(y-x)}\,\rho^{A_1\cdots A_{q-1}B_1\cdots B_rA_qB_{r+1}\cdots B_p}(x)
\end{aligned} \tag{129}$$

5.8 Short Distance Expansion for $\rho^{A_1\cdots A_q}(y)\,O(x)$

Our last contribution to the expansions for the coadjoint representation is given below. Special cases are explicitly written and supercede the more general expressions.

$$R^{\mathrm{J}_1\mathrm{J}_2}_{r^{\mathrm{J}_1\mathrm{J}_2}} * \bar{U}^{\mathrm{M}}_{\omega^{\mathrm{M}}} = \bar{G}^{\mathrm{J}_1}_{(r^{\mathrm{J}_1\mathrm{J}_2}\,\omega^{\mathrm{M}})}\,\delta^{\mathrm{MJ}_2} - \bar{G}^{\mathrm{J}_2}_{(r^{\mathrm{J}_1\mathrm{J}_2}\,\omega^{\mathrm{M}})}\,\delta^{\mathrm{J}_1\mathrm{M}} \tag{130}$$

$$R^{\mathrm{J}_1\mathrm{J}_2}_{(r^{\mathrm{J}_1\mathrm{J}_2})} * \bar{U}^{\mathrm{LMN}}_{\omega^{\mathrm{LMN}}} = \tfrac{1}{2}\delta^{\mathrm{J}_1\mathrm{L}}\,\delta^{J_2M}\,\bar{G}^{\mathrm{N}}_{(r'\omega+r\omega')} - 2\delta^{\mathrm{J}_2\mathrm{L}}\,\delta^{\mathrm{MN}}\,\bar{G}^{\mathrm{J}_1}_{(r\omega')} + 2\,\delta^{\mathrm{J}_1\mathrm{L}}\,\delta^{\mathrm{MN}}\,\bar{G}^{\mathrm{J}_2}_{(r\,\omega')} \tag{131}$$

$$R^{\mathrm{I_1\,I_2}}_{r^{\mathrm{I_1 I_2}}} * \bar{G}^{\mathrm{K}}_{\psi^{\mathrm{K}}} = \tfrac{1}{2}\delta^{\mathrm{K}[\mathrm{I_2}}\,\bar{U}^{\mathrm{I_1}]}_{(r^{\mathrm{I_1 I_2}}\psi^{\mathrm{K}})} - \tfrac{1}{2}\big(\bar{G}^{\mathrm{J_1}}_{(r^{\mathrm{J_1 J_2}}\,\psi^{\mathrm{L}})}\,\delta^{\mathrm{J_2\,L}} - \bar{G}^{\mathrm{J_2}}_{(r^{\mathrm{J_1 J_2}}\psi^{\mathrm{L}})}\,\delta^{\mathrm{J_1 L}}\big) \tag{132}$$

$$\begin{aligned}
R^{\mathrm{J_1\cdots J_p}}_{r\{J_p\}} * \bar{U}^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_{\mathrm{m}}}_{\omega\{V_m\}} &= -\tfrac{1}{2}i(i)^{\{[\frac{p}{2}]-[\frac{p+2}{2}]\}}\,\delta^{[\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_{\mathrm{m-2}}}_{[\bar{\mathrm{J}}_1\cdots\bar{\mathrm{J}}_{\mathrm{p}}]}\,\delta^{\bar{\mathrm{V}}_{\mathrm{m-1}}],\bar{\mathrm{V}}_{\mathrm{m}}}\,\delta^{\mathrm{m,p+2}}\,\bar{L}_{(r\omega)''} \\
&+ i^{\{[\frac{p}{2}]-[\frac{p+1}{2}]\}}\delta^{\mathrm{p+1,m}}\bar{G}^{\bar{\mathrm{V}}_{\mathrm{m}}}_{(r\omega)'}\,\delta^{[\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_{\mathrm{m-1}}]}_{[\mathrm{J_1\cdots J_p}]} \\
&+ 2i(-1)^{p+1}\delta^{\mathrm{p+3,m}}\,(i)^{[\frac{p}{2}]-[\frac{p+2}{2}]}\,\bar{G}^{[\bar{\mathrm{V}}_1}_{(r\omega)''}\,\delta^{\bar{\mathrm{V}}_2\cdots\bar{\mathrm{V}}_{\mathrm{m-2}}}_{[\mathrm{J_1\cdots J_p}]}\,\delta^{\bar{\mathrm{V}}_{\mathrm{m-1}},]\bar{\mathrm{V}}_{\mathrm{m}}} \\
&+ i(-1)^p\bar{T}^{\bar{\mathrm{V}}_1[\bar{\mathrm{V}}_{\mathrm{m}}}_{(r\omega)'}\delta^{\bar{\mathrm{V}}_2\cdots\bar{\mathrm{V}}_{\mathrm{m-1}}]}_{[\mathrm{J_1\cdots J_p}]}\delta^{p+2,m} \\
&+ (-1)^{\mathrm{pm}}(i)^{\{[\frac{m}{2}]+[\frac{p}{2}]-[\frac{m+p-2}{2}]\}}\,\bar{U}^{\bar{\mathrm{V}}_{\mathrm{p}}+1\cdots\bar{\mathrm{V}}_{\mathrm{m-p}}}_{(r\omega)'}\delta^{\bar{\mathrm{V}}_1\cdots\bar{\mathrm{V}}_{\mathrm{p}}}_{[\mathrm{J_1\cdots J_p}]}\quad ,
\end{aligned} \tag{133}$$

$$\begin{aligned}
R^{\mathrm{J_1\cdots J_p}}_{r\{J_p\}} * \bar{R}^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{m}}}_{\rho\{T_m\}} &= \delta^{[\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{m}}]}_{[\mathrm{J_1\cdots J_p}]}\,\delta^{\mathrm{p\,m}}\,\bar{L}_{(-(\frac{p}{2}-2)r'\rho-(\frac{p}{2}-1)\,r\rho')} \\
&+ (-1)^p\,\{2(i)\bar{G}^{[\bar{\mathrm{T}}_1}_{((2-p)r'\rho-(p-1)r\rho')}\,\delta^{\bar{\mathrm{T}}_2\cdots\bar{\mathrm{T}}_{\mathrm{m}}]}_{[\mathrm{J_1\cdots J_p}]}\delta^m_{p+1} \\
&+ (i)(i)^{[\frac{p}{2}]-[\frac{p-1}{2}]}\sum_{r=1}^{p}(-1)^r\bar{G}^{\mathrm{J_r}}_{(r\rho)}\delta^{\mathrm{m}}_{\mathrm{p-1}}\delta^{[\mathrm{J_1\cdots J_{r-1}J_{r+1}\cdots J_p}]}_{[\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{r-1}}\bar{\mathrm{T}}_{\mathrm{r}}\ldots\bar{\mathrm{T}}_{\mathrm{m}}]}\} \\
&+ \sum_{r=1}^{p}(-1)^{r+1}2\bar{T}^{[\bar{\mathrm{T}}_1\,|\mathrm{J_r}|}_{(r\rho)}\delta^{\bar{\mathrm{T}}_2\cdots\bar{\mathrm{T}}_{\mathrm{m}}]}_{[\mathrm{J_1\cdots J_{r-1}\,J_{r+1}\cdots J_p}]}\delta^p_m \\
&- \sum_{r=1}^{p}(-1)^{r-1}\bar{U}^{[\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{m-p+1}}\,|\mathrm{J_r}|}_{(r\rho)}\delta^{\bar{\mathrm{T}}_{\mathrm{m-p+2}}\cdots\bar{\mathrm{T}}_{\mathrm{m}}]}_{\mathrm{J_1\cdots J_{r-1}J_{r+1}\cdots J_p}} \\
&+ i^{\{[\frac{p}{2}]+[\frac{m-p}{2}]-[\frac{m}{2}]\}}\,\bar{R}^{[\bar{\mathrm{T}}_{\mathrm{p+1}}\cdots\bar{\mathrm{T}}_{\mathrm{m}}}_{(2r'\rho+r\rho')}\delta^{\bar{\mathrm{T}}_1\cdots\bar{\mathrm{T}}_{\mathrm{p}}]}_{[\mathrm{J_1\cdots J_p}]}.
\end{aligned} \tag{134}$$

Typical operator product expansions are then $(q>2)$:

$$\begin{aligned}
\rho^{A_1\cdots A_q}(y)\,D(x) &= \frac{(\frac{p}{2}-2)}{2\pi i(y-x)}\left(\frac{\rho^{A_1\cdots A_q}(x)}{(y-x)} - (\tfrac{p}{2}-1)\partial_x\rho^{A_1\cdots A_q}(x)\right) \\
&- \frac{(i)^{[\frac{p}{2}]-[\frac{p+2}{2}]}}{2\pi(y-x)}\left(\frac{\omega_A^{A_1\cdots A_qA}(x)}{(y-x)^2} + \frac{\partial_x\omega_A^{A_1\cdots A_qA}(x)}{(y-x)} + \tfrac{1}{2}\partial_x^2\omega_A^{A_1\cdots A_qA}(x)\right)
\end{aligned} \tag{135}$$

$$\begin{aligned}
\rho^{A_1\cdots A_q}(y)\psi^B(x) &= \frac{i^{[\frac{q}{2}]-[\frac{q+1}{2}]}}{2\pi i(y-x)}\left(\partial_x\,\omega^{A_1\cdots A_qB}(x) - \frac{1}{(y-x)}\omega^{A_1\cdots A_qB}(x)\right) \\
&+ \frac{2(-1)^{q+1}(i)^{[\frac{q}{2}]-[\frac{q+2}{2}]}}{\pi(y-x)}\left(\frac{\omega_A^{BA_1\cdots A_qA}(x)}{(y-x)^2} + \frac{\omega_A^{BA_1\cdots A_qA}(x)}{(y-x)} + \tfrac{1}{2}\omega_A^{BA_1\cdots A_qA}(x)\right) \\
&+ (-1)^q\left(\frac{(2-p)\rho^{BA_1\cdots A_q}(x)}{\pi(y-x)^2} - \frac{(p-1)\partial_x\rho^{BA_1\cdots A_q}(x)}{\pi(y-x)}\right) \\
&+ \frac{(i)^{[\frac{q}{2}]-[\frac{q-1}{2}]}}{2\pi(y-x)}\sum_{r=1}^{q}\delta^{A_rB}\,\rho^{A_1\cdots\hat{A}_r\cdots A_q}(x)
\end{aligned} \tag{136}$$

$$\begin{aligned}
\rho^{A_1\cdots A_q}(y)\,A^{B_1B_2}(x) &= (i)^q\left(\frac{\omega^{[B_1|A_1\cdots A_q|B_2]}}{2\pi(y-x)^2} + \frac{\partial_x\omega^{[B_1|A_1\cdots A_q|B_2]}}{2\pi(y-x)}\right) \\
&+ \sum_{r=1}^{p}(-1)^{r+1}\frac{-1}{\pi i(y-x)}\rho^{B_2A_1\cdots A_{r-1}B_1A_{r+1}\cdots A_q}
\end{aligned} \tag{137}$$

$$
\begin{aligned}
\rho^{A_1\cdots A_q}(y)\,\omega^{B_1\cdots B_n}(x) &= (-1)^{pn}(i)^{[\frac{n}{2}]+[\frac{p}{2}]-[\frac{n+p-2}{2}]}\left(\frac{\omega^{A_1\cdots A_qB_1\cdots B_n}(x)}{2\pi i(y-x)^2}+\frac{\partial_x\omega^{A_1\cdots A_qB_1\cdots B_n}(x)}{2\pi i(y-x)}\right)\\
&- \sum_{r=1}^{p}(-1)^{r-1}\frac{1}{2\pi i(y-x)}\rho^{B_1\cdots B_{n-r+1}A_rB_{n-r+2}\cdots B_nA_1\cdots\hat{A}_r\cdots A_q}(x) \qquad (138)\\
\rho^{A_1\cdots A_q}(y)\,\rho^{B_1\cdots B_n}(x) &= i^{[\frac{q}{2}]+[\frac{n-2q}{2}]-[\frac{n-p}{2}]}\left(\frac{\rho^{A_1\cdots A_qB_1\cdots B_n}(x)}{\pi i(y-x)^2}+\frac{\partial_x\rho^{A_1\cdots A_qB_1\cdots B_n}(x)}{\pi i(y-x)}\right) \qquad (139)
\end{aligned}
$$

6 Conclusions

This work shows that the dual representation of an algebra provides a natural definition of a short distance expansion or operator product expansion. The operators are not built from an enveloping algebra as in the Sugawara constructions [30] but instead exists in their own right. For example in the section 4.1 the operators, $D(x)$, satisfy the same operator product expansions as one has in Virasoro theories [12] which is built from a lattice. However, notice the $A(x)$ that appears in our discussion of affine Lie algebras does not appear in conformal field theory treatments of WZNW [32, 35, 25] where the energy-momentum tensor is constructed from the current algebra. Our construction is otherwise model independent and can be implemented for *any* algebra that admits a dual. Underlying free-field theories are *not* necessary to define these expansions. Thus, the roles of symmetry groups in defining these expansions is moved firmly to the foreground. The origin of this model independence is that we exploit the natural bifurcation of the initial data where the algebraic elements serve as conjugate momenta and the dual of the algebra serves as the conjugate coordinates. This feature is shared by the symplectic structures of actions like Yang-Mills theory and N-extended affirmative actions [16, 4]. The coadjoint representation enforces a short distance expansion already at the level of the Poisson bracket of the elements with no Hamiltonian required.

Another implication that we wish to note is that since we never relied on any specific model in our discussion, this implies that at no point was it necessary to invoke Wick rotations to a Euclideanized formulation to justify the form of the short-distance expansions. We note that just as in the case of Virasoro theories and some other conformal field theory based operators, the central extension can be further restricted by demanding unitarity. Choices of the central extension can be determined through the Kac determinant [18, 13, 11].

This viewpoint has been applied to the model-independent N-extended supersymmetric $\mathcal{GR}$ Virasoro algebra [15] to obtain for the first time its representation in terms of short distance expansions. The complete set of such expansions have been presented in the fifth section of this work. This success is expected to open up further avenues of study. Since these short distance expansions are now known for arbitrary values of N, this means that our new results may be used to study the possible existence of 1D, $N = 16$ or $N = 32$ supersymmetric NSR-type models. The question of whether such an approach can lead to a new manner for probing M-theory is now a step closer to being answered.

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